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MATHEMATICS SIMPLIFIED

PRACTICE SERIES FOR

CLASS 12th, JEE PRE & MAINS

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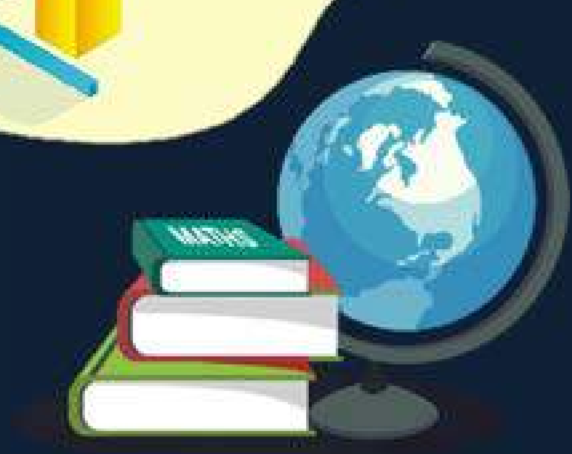

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Preface

Bringing up the series ***"Mathematics Simplified - Practice Series for Class 12th, JEE PRE & Mains"***, carefully revised with the session wise theory and exercises, aims to help candidates to learn & tackle the mathematical problems. The book has 28 Chapters covering the whole syllabus for the JEE Pre and Mains as prescribed. Each chapter is divided into sessions giving complete clarity to concepts. Apart from concise session wise theory, it contains a huge amount of questions that are provided in every chapter under Practice Problems. Prepared under great expertise, it is a highly recommended practice-book to develop a strong base for best performance in JEE and various engineering entrances.

Every effort has been made to eliminate printing or principle errors. Still, we apologise for any inaccuracies that may have snuck into the book unnoticed. Teachers, students, and any readers who have suggestions for improving the work would be greatly appreciated.

Authors

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01 | Relation

Relation

A relation R from set X to set Y is defined as a subset of the cartesian product $X \times Y$ i,e $R \subseteq X \times Y$.

Domain and range of a relation

Domain The set of first elements of all ordered pairs in R i.e, $\{x:(x, y) \in R\}$ is called the domain of relation R.

Range The set of second elements of all ordered pairs in R i.e, $\{y:(x, y) \in R\}$ is called the range of relation R.

Types of relation

1. **Void (or empty) Relation:** Let X be a set. Then, $\varphi \subseteq X \times X$ and so it is a relation on X. This relation is called the void or empty relation on set X.

   i.e. $R = \varphi \subseteq X \times X$
2. **Universal Relation:** A relation R in a set X is called universal relation, if each element of X is related to every element of X, i.e. $R = X \times X$
3. **Reflexive Relation:** A relation R defined on a set A is said to be reflexive , if $(x,x) \in R, \forall x \in A$ or $\text{xRx}, \forall \text{x} \in \text{A}$.
4. **Symmetric Relation:** A relation R defined on a set A is said to be symmetric, if $(x, y) \in R \Rightarrow, (y, x) \in R \quad \forall x, y \in A$ or $\text{xRy} \Rightarrow \text{yRx}, \forall \text{x}, \in \text{A}$.
5. **Transitive Relation:** A relation R defined on a set A is said to be transitive, if $(x, y) \in R$ & $(y, z) \in R \Rightarrow (x, z) \in R$ or $\text{xRy}, \text{yRz} \Rightarrow \text{xRz}, \forall \text{x,y,z} \in A$

Practice Problems

1) Show that the relation R on the set A = {1, 2, 3} given by R = {(1, 1), (2, 2) (3, 3), (1, 2), (2, 3)} is reflexive but neither symmetric nor transitive.
2) Show that the relation R on the set A = {1, 2, 3} given by R = {(1, 2), (2,1), is symmetric but neither reflexive nor transitive.
3) Check the following relations R and S for reflexivity, symmetric and transitivity:

   i) aRb if b is divisible by a, a, $b \in N$.

   ii) I_1 S I_2 if $I_1 \perp I_{2,}$, where I_1 and I_2 are straight lines in a plane.
4) Determine whether each of the following relations are reflexive, symmetric and transitive:
5) Relation R on the set A = {1, 2, 3,..............,13, 14} defined as R = {(x, y) : 3x – y = 0}
6) Show that the relation R on R defined as R = {(a, b) : $a \le b$ }, is reflexive and transitive but not symmetric.
7) Let X = {1, 2, 3, 4, 5, 6, 7, 8, 9}, Let R_1 be a relation on X given by $R_1 = \{x, y\} : x - y$ is divisible by 3} and R_2 be another relation on X given by $R_2 = \{(x, y) : (x, y) \subset (1,4,7)$ $or (x, y) \subset (2,5,8)$ $or (x, y) \subset (3,6,9)\}$. Show that $R_1 = R_2$.
8) Show that the relation R on the set R of all real numbers, defined as $R = \{(a,b) : a \le b^2\}$ is neither reflexive nor symmetric nor transitive.
9) Let A = {1, 2, 3}. Then, show that the number of relations containing (1, 2) and (2, 3) which are reflexive and transitive but not symmetric is there.

10) Let A be the set off all human beings in a town at a particular time. Determine whether each of the following relation are reflexive, symmetric and transitive:
11) R = {(x, y) : x is father of y}
12) Check whether the relation R defined on set A = {1, 2, 3, 4, 5, 6} as R = {(a, b) : b = a + 1}
13) Check whether the relation R on R defined by R = $\{(a,b): a \le b^3\}$ is reflexive, symmetric or transitive.
14) Let R be a relation defined on the set of natural numbers N as R = $\{(x,y):(x,y \in N, 2xy = 41\}$
15) Give an example of a relation which is, reflexive and symmetric but not transitive.

Equivalence Relation

Definition

A relation R defined on a set A is said to be an equivalence relation on A if it is

i) Reflexive i.e. $(a,a) \in R$ for all $a \in A$.
ii) Symmetric i.e. $(a,b) \in R \Rightarrow (b,a) \in R$ for all $a,b \in A$.
iii) Transitive i.e. $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$ for all $a,b,c \in A$.

Practice Problems

1) Let R be a relation on the set of all lines in a plane defined by $(I_1, I_2) \in R \Leftrightarrow line\ I_1$ is parallel to line $I_{2.}$ Show that R is an equivalence relation.
2) Show that the relation 'is congruent to' on the set of all triangles in an equivalence relation.

3) Show that the relation R defined on the set A of all triangles in a plane as R = {(T_1, T_2) : T_1 is similar to T_2} is an equivalence relation.

4) Let n be a positive integer. Price that the relation R on the set Z of all integers numbers defined by $(x, y) \in R \Leftrightarrow x - y$ is divisible by n, is an equivalence of relation on Z.

5) Show that the relation R on the set A of all the books in a library of a college given by R = {(x, y) : x and y have same number of pages}, is an equivalence relation.

6) Show that the relation R on the set A = {1, 2, 3, 4, 5}, given by R = $\{(a,b): |a-b| \text{ is even}\}$, is an equivalence relation.

7) Show that the relation R on the set of $A = \{x \in Z : 0 \le x \le 12\}$, given by $R = \{(a,b): |a-b|\}$ is a multiple of 4} is an equivalence relation. Find the set of all elements related to 1 i.e. equivalence class [1].

8) Show that the relation R on the set of A of points in a plane, given by R = {(P, Q): Distance of the point P from the origin is same as the distance of the point Q from the origin}, is an equivalence relation. Further show that the set of all points related to a point $P \ne (0,0)$ is the circle passing through P with origin as centre.

9) Prove that the relation R on set N × N defined by $(a,b)R(c,d) \Leftrightarrow a+b=b+c$ for all $(a,b),(c,d) \in N \times N$ is an equivalence relation.

10) Let A = {1, 2, 3…………,9} and R be the relation on A × A defined by $(a,b)R(c,d)$ *if* a+b=b+c for all $(a,b),(b,c) \in A \times A$. Prove that R is an equivalence relation and also obtain the equivalence class [(2, 5)].

11) Let N be the set of all natural numbers and let R be a relation on N × N, defined by (a, b) R (c, d) ⇔ad = bc for all (a, b), (c, d) ∈N × N.

12) show that R is an equivalence relation on N × N. Also, find the equivalence class [(2, 6)].

13) Let N denote the set of all natural numbers and R be the relation on N × N defined by $(a,b)R(c,d) \Leftrightarrow ab(b+c)=bc(a+d)$. check whether R is an equivalence relation on N × N.

14) Show that the number of equivalence relation on the set {1, 2, 3} containing (1, 2) and (2, 1) is two.

15) Given a non-empty set X, consider P(X) which is the set of all subsets of X. Define a relation in P(X) as follows:

16) For subsets A, B in P(X), A R B if $A \subset B$. Is R an equivalence relation on P(X)? justify your answer.

17) Let R be the equivalence relation in the set A = {0, 1, 2, 3, 4, 5} given by R = {(a, b) : 2 divides (a – b)}. Write the equivalence class [0].

18) If R is an equivalence relation on a set A, then is also an equivalence relation on A.

19) Or

20) The inverse of an equivalence relation is an equivalence relation.

21) Show that the relation R defined by R = {(a, b) : a – b is divisible by 3; a, $b \in Z$} is an equivalence relation.

22) Show that the relation R on the set Z of integers, given by R = {(a, b) : 2 divides a – b}, is an equivalence relation.

23) Prove that the relation R on Z defined by $(a,b) \in R \Leftrightarrow a\text{-}b$ is divisible by 5 is an equivalence relation on Z.

24) Let R be a relation on the set A of ordered pairs of integers defined by (x, y) $R(u,u)$ if xu=yu Show that R is an equivalence relation.

25) Show that the relation R defined on the set A of all polygons as $R = \{(P_1, P_2) : P_1$ and P_2 have same number of sides$\}$, is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides 3, 4 and 5?

26) Let R be the relation defined on the set A = {1, 2, 3, 4, 5, 6, 7} by R = {(a, b) : both a and b are either odd or even}. Show that R is an equivalence relation. Further, show that all the elements of the subset {1, 3, 5, 7} are related to each other and all the elements of the subset {2, 4, 6} are related to each other, but no element of the subset {1, 3, 5, 7} is related to any element of the subset {2, 4, 6}.

27) Write the smallest reflexive relation on set A = {1, 2, 3, 4}.

28) If A = {3, 5, 7} and B = {2, 4, 9} and R is a relation given by "is less than", write R as a set ordered pairs.

29) State the reason R on the set {1, 2, 3} given by R = {(1, 2), (2,1) not to be transitive.

30) Let $R = \{(a, a^3) :$ a is a prime number less than 5$\}$ be a relation. Find the range of R.

31) Let R be the equivalence relation on the set Z of integers given by R = {(a, b) : 2 divides a – b}. write the equivalence class [0].

——==00==——

02 Functions

Recapitulation

Function as a set of ordered pairs

Let A and B be two non – empty sets. A relation f from A to B i.e. a sub set of A × B is called a function (or a mapping or a map) from A to B, if

(i) For each $a \in A$ there exists $b \in B$ such that $(a,b) \in f$.

(ii) $(a,b) \in f$ and $(a,c) \in f \Rightarrow b \Rightarrow c$

Domain, Codomain and Range of a Function

The element of X are called the **domain** of f and the element of Y are called the **codomain** of f. The images of the element of X is called the **range** of f which is a subset of Y.

Types Of Function

1. **One-one (or Injective) and Many-one Function**

   A function $f : X \to Y$ is said to be a one-one function, if the images of distinct elements of X under f are distinct, thus f is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for all $x_1,\ x_2 \in$ X.

   A function which is not one-one, is known as **many-one** function

2. **Onto (or Surjective) and Into Function**

   A function $f : X \to Y$ is said to be an onto function, if every element of Y is images of some elements of X under f.

   I,e. for every $y \in Y$, $\exists$ an element x in X such that f (x) = y.

   In other words a function is called an onto function, if its range is equal to codomain.

A function $f: X \to Y$ is said to be into function if $\exists$ atleast one element in Y, which do not have any pre image in X.

3. Bijective Function

A function $f: X \to Y$ is said to be a bijective function, if it is both one-one and onto.

Practice Problems

1) Show that the function $f: N \to N$ given by $f(1) = f(2) = 1$ and $f(x) = x - 1$ for every $x \geq 2$, is onto but not one – one.

3) Show that the Signum function $f: R \to R$, given by

$$f(x) = \begin{cases} 1 & if\ x > 0 \\ 0 & if\ x = 0 \\ -1 & if\ x < 0 \end{cases}$$ is neither one – one nor onto.

4) Show that the function $f: N \to N$ given by $f(x) = 2x$, is one-one but not onto.

5) Prove that $f: R \to R$, given by $f(x) = 2x$, is one-one and onto.

6) Show that the function $f: R \to R$ defined as $f(X) = X^2$, is neither one-one nor onto.

7) Show that $f: R \to R$ defined as $f(X) = X^3$, is a bijection.

8) Show that the function $f: R_0 \to R_0$, defined as $f(X) = \frac{1}{X}$, is one-one onto, where R_0 is the set of all non-zero real numbers. Is the result true, if the domain R_0 is replaced by N with co-domain being same as R_0?

9) Prove that the greatest integer function $f: R \to R$, given by $f(X) = [X]$, is neither one-one nor onto, where [x] denotes the greatest integer less or equal to x.

10) We observe that $f(-2) = f(2)$. So, f is not one-one.

11) Show that the function $f: R \to R$ defined as $f(X) = ax + b$, where $a, b \in R$, $a \neq 0$ is a bijection.

12) Show that the function $f: R \to R$ given by $f(X) = \cos x$ for all $a \in R$. is neither one-one nor onto.

13) Let A and B be two sets. Show that $f\ A \times B \to B \times A$ defined by $f(a,b) = (b,a)$ is a bijection.

14) Consider the identify function $I_n : N \to N$ defined as, $I_n(x) = x$ for all $x \in N$. Show that although I_n is onto but $I_n + I_n : N \to N$ defined as $(I_n + I_n)x = I_n(x) = I_n(x) = x + x = 2x$ is not onto.

15) Consider the function $f: \left[0, \frac{\pi}{2}\right] \to R$ given by $f(x) = \sin x$ and $g: \left[0, \frac{\pi}{2}\right] \to R$ given by $g(x) = \cos x$. show that f and g are one-one, but $f + g$ is not one-one.

16) Let $f: X \to Y$ be a function. Defined a relation R on X given by $R = \{(a,b) : f(a) = f(b)\}$. Show that R is an equivalence relation on X.

17) Show that the function $f: R \to \{x \in R : -1x < 1]$ defined by $f(x) = \frac{x}{1+|x|}, x \in R$ is one-one onto function.

18) Show that $f : N \to N$ defined by $f(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if n is even} \end{cases}$

is many – one onto function.

19) Show that the function $f : N \to N$ given by, $f(n) = n - (-1)^n$ for all $n \in N$ is a bijection.

20) $f : R \to R$, defined by $f(x) = \frac{x}{x^2 + 1}$

21) Show that the function $f : R - (3) \to R(1)$ given by

$f(x) = \frac{x-2}{x-3}$ is a bijection.

22) Let A = [-1, 1]. Then, discuss whether the following functions from A to itself are one-one, onto or bijective:

23) If $f\ R \to R$ be the function defined by $f(x) = 4x^3 + 7$, show that f is a bijection.

24) Show that the logarithmic function $f : R_0^+ \to R$ given by $f(x) = \log_a x,\ a > 0$ is a bijective.

25) If A = {1, 2, 3}, show that a one-one function $f : A \to A$ must be onto.

26) If A = {1, 2, 3}, show that an onto function $f : A \to A$ must be one-one.

27) Find the number of all onto functions from the set A = {1, 2, 3,…….,n} to itself.

28) Let $f: N \to N$ be defined by $f(n) = \begin{cases} n+1 & if\ n\ is \text{ odd} \\ n-1 & if \text{ n is even} \end{cases}$ show that is a bijection.

Composition Of Functions
Definition

Let $f: A \to B$ and $g: B \to C$ be two functions. Then a function $gof: A \to C$ defined by $(gof)(x) = g(f(x))$, for all $x \in A$

1) Let $f:\{2,3,4,5\} \to \{3,4,5,9\}$ and $g:\{3,4,5,9\} \to \{7,11,15\}$ be functions defined as $f(2)=3, f(4)=f(5)=5$ and, $g(3)=g(4)=7$ and $g(5)=g(9)=11$.

2) Let $f:\{1,3,4\} \to \{1,2,5\}$ and $g:\{1,2,5\} \to \{1,3\}$ be given by $f=\{(1,2),(3,5),(4,1)\}$ and $g=\{(1,3),(2,3),(5,1)\}$. Write down gof.

3) Find gof and fog, if $f: R \to R$ and $g: R \to R$ are given by $f(x) = |x|$ and $g(x) = |5x-2|$.

4) If the function $f: R \to R$ be given by $f(x) = x^2 + 2$ and $g: R \to R$ be given by $g(x) = \frac{x}{x-1}$. Find fog and gof.

5) If $f: R - \left\{\frac{7}{5}\right\} \to R - \left\{\frac{3}{5}\right\}$ be defined as $f(x) = \frac{3x+4}{5x-7}$ and $g: R - \left\{\frac{3}{5}\right\} \to R - \left\{\frac{7}{5}\right\}$ be defined as $g(x) = \frac{7x+4}{5x-3}$. Show that $gof = I_A$ and $fog = I_B$, where $B = R - \left\{\frac{3}{5}\right\} and A = R - \left\{\frac{7}{5}\right\}$.

6) If $f:R\to R$ is defined by $f(x)=x^2-3x+2$, find $f(f(x))$.

7) If $f(x)=x^2+3x+1,\ g(x)=2x-3$ $f,g:R\to R$ are defined respectively by $f(x)=x^2+3x+1,\ g(x)=2x-3$, find

8) fog (ii) gof (iii) fof (iv) gog.

9) Let f,g and h be function from R to R. show that:

10) $(f+g)oh=foh+goh$ (ii) $(fg)oh=(foh)(goh)$

11) Let $f:R\to R$ be the signum function defined as $f(x)=\begin{cases}1 & if\ x>0\\ 0 & if\ x=0\\ -1 & if\ x<0\end{cases}$ and $g:R\to R$ be the greatest integer function given by $g(x)=[x]$. Then prove that fog and gof coincide in [-1, 0].

12) Let $A=\{x\in R:0\le x\le 1\}$. If $f:A\to A$ is defined by $f(x)=\begin{cases}x & if\ \text{x}\in \text{Q}\\ 1-x & if\ \text{x}\in Q\end{cases}$ then prove that $fof(x)=x$ for all $x\in A$.

13) Let $f:R\to R$ and $g:R\to R$ be two functions such that $fog(x)=\sin x^2$ and $gof(x)=\sin^2 x$. Then, fund $f(x)$ and $g(x)$.

Properties Of Composition of Functions

1) The composition of functions is not commutative i.e. $fog\ne gof$.
2) The composition of functions is associative i.e. if f,g,h are three functions such that $(fog)oh$ and (goh) exist, then $(fog)oh=fo(goh)$.

3) The composition of two bijections is i.e. if f and g are two bijections, then gof is also a bijection.
4) Let $f : A \to B,\ g : B \to A$ be two functions such that $gof = I_A$. Then, f is an injection and g is a surjection.
5) Let $f : A \to B$ and $g : B \to A$ be two function such that $fog == I_g$. Then, f is a surjection and g is an injection.
6) Consider $f : N \to N$, $g : N \to N$ defined as $f(x) = 2x,$ $g(y) = 3y + 4$ and $h(z) = \sin z$ for all $x, y, z \in N$. Show that $ho(gof) = (hog)of$.
7) Give examples of two functions $f : N \to N$ and $g : N \to N$ such that gof is onto but f is not onto.
8) Give examples of two functions $f : N \to N$ and $g : N \to N$ such that gof is injective but g is not injective.

Composition Of Real Functions

Defination

Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then,

$gof : x = \{x \in D_1 : f(x) \in D_2\} \to R$ and,

$fog\ \mathrm{y} = \{x \in D_2 : g(x) \in D_1\} \to R$ are defined as

$gof(x) = g(f(x))$ for all $x \in X$ and $fog(x) = f(g(x))$ for all $x \in Y$.

Practice Problems

1. If $f : R \to R$ and $g : R \to R$ be functions defined by $f(x) = x^2 + 1$ and $g(x) = \sin x$, then find fog and gof.
2. If $f(x) = e^x$ and $g(x) = \log_e x (x > 0)$, find fog and gof. Is $fog = gof$?

3. If $f(x)=\sqrt{x}\ (x>0)$ and $g(x)=x^2-1$ are two real functions, find fog end gof. Is $fog=gof$?

4. Let f and g be real functions defined by $f(x)=\frac{x}{x+1}$ and $g(x)=\frac{1}{x+3}$. Describe the functions gof and fog (if they exist).

5. If $f(x)=\frac{3x-2}{2x-3}$, prove that $f(f(x))=\frac{2x+1}{2x+3}$ for all $x\in R-\left\{\frac{3}{2}\right\}$.

6. If $f(x)=\frac{1}{2x+1}$, $x\neq\frac{1}{2}$ then show that $f(f(x))=\frac{2x+1}{2x+3}$, provided that $x\neq-\frac{1}{2},-\frac{3}{2}$.

7. If $f(x)=\frac{x-1}{x+1}$, $x\neq-1$, then show that $f(f(x))=-\frac{1}{x}$ provided that $x\neq0,-1$.

8. If the function $f:R\to R$ be defined by $f(x)=x^2+5x+9$, find $f^{-1}(8)$ and $f^{-1}(9)$.

9. If the function $f:C\to C$ be defined by $f(x)=x^2-1$, find $f^{-1}(-5)$ and $f^{-1}(-8)$.

Inverse Of a Function

Definition

Let $f: A \to B$ be a bijection. Then a function $g: B \to A$ which associates each element $y \in B$ to a unique element $x \in A$ such that $f(x) = y$ is called the inverse of f.

i.e. $f(x) = y \Leftrightarrow g(y) = x$ The inverse of f is generally denoted by f^{-1}

Thus, if $f: A \to B$ is a bijection, then $f^{-1}: B \to A$ is such that $f(x) = y \Leftrightarrow f^{-1}(y) = x$

Algorithm

Let $f: A \to B$ be a bijection. To find the inverse of we following steps:

Step I Put $f(x) = y$, where $y \in B$ and $x \in A$.

Step II Solve $f(x) = y$ to obtain x in terms of y.

Step III In the relation obtained in step II replace x by $f^{-1}(y)$ to obtain the required inverse of f.

Following examples will illustrate the above algorithm.

1. Let S = {1, 2, 3}. Determine whether the function $f: S \to S$ defined as below have inverse. Find f^{-1}, if it exists.
2. Consider $f: \{1,2,3\} \to \{a,b,c\}$ given by $f(1) = a, f(2) = b$ and $f(3) = c$. Find the inverse $\left(f^{-1}\right)^{-1} = f$.

3. Let $f : N \cup \{0\} \to N \cup \{0\}$ be defined by $f(n) = \begin{cases} n+1 & if\ n\ is\ even \\ n-1 & if\ n\ is\ odd \end{cases}$ show that f is invertible and $f = f^{-1}$

Properties Of Inverse Of a Function

1. The inverse of a bijection is unique.
2. The inverse of a bijection is also a bijection.
3. If $f : A \to B$ is a bijection and $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A and I_B are the identity functions on the set A and B respectively.
4. If $f : A \to B$ and $g : B \to C$ are two bijections, then $gof : A \to C$ is a bijection and $(gof)^{-1} = f^{-1}og^{-1}$.
5. Let $f : A \to B$ and $g : B \to A$ be two functions such that $gof = I_A$ and $fog = I_B$. Then, f and g are bijections and $g = f^{-1}$.
6. Let $f : A \to B$ be an invertible function. Show that the inverse of f^{-1} is f. i.e., $\left(f^{-1}\right)^{-1} = f$.

Algorithm

Step I Obtain the function and check its bijectivity.

Step II If f is a bijection, then it is invertible. In order to find the increase of f, put $fof^{-1}(x) = x \Rightarrow f(f^{-1}(x)) = x$.

Step III Use the formula for $f(x)$ and replace x by f^{-1} in it to obtain the LHS of $f(f^{-1}(x)) = x$. Solve this equation for $f^{-1}(x)$ to get $f^{-1}(x)$.

Following examples will illustrate the above algorithm:

1. Show that $f:[-1,1]\rightarrow R$, given by $f(x)=\dfrac{x}{x+2}$ is one-one. Find the inverse of the function $f:[-1,1]\rightarrow range\ (f)$.
2. Let $f:R\rightarrow R$ be defined as $f(x)=10x+7$. Find the function $g:R\rightarrow R$ such that $gof = fog = I_R$.
3. Let $f:N\rightarrow Y$ be a function defined as $f(x)=4x+3$, where $Y=\{y\in N: y=4x+3 \text{ for same } x\in N\}$. show that f is invertible. Find is inverse.
4. Let $Y=\{n^2: n\in N\}\subset N$. Consider $f:N\rightarrow Y$ given by $f(n)=n^2$ Show that f is invertible find the inverse of f.
5. Let $f:N\rightarrow R$ be a function defined as $f(x)=4x^2+12x+15$. Show that $f:N\rightarrow range\ (f)$ is invertible. Find the inverse of f.
6. Consider $f:\{1,2,3\}\rightarrow\{a,b,c\}$ and $g:\{a,b,c\}\rightarrow\{apple,\ ball,\ cat\}$ defined as $f(1)=a,\ f(2)=b$ $f(3)=c$, $g(a)=apple$, g(b)=*ball* and g(c)=*cat*. Show that f, g and gof are invertible. Find f^{-1}, g^{-1} and (gof)$^{-1}$ $=f^{-1}og^{-1}$.
7. Consider $f:R\rightarrow R$ given by $f(x)=4x+3$. Show that f is invertible. Find the inverse off f.
8. Consider $f:R'\rightarrow R[4,\infty)$ given by $f(x)=x^2+4$ Show that f is invertible with inverse f^{-1} of given by $f^{-1}(x)=\sqrt{x-4}$, where R^+ is the set of all non-negative real numbers.

9. If $f(x)=\frac{4x+3}{6x-4}, x\neq\frac{2}{3}$, show that $fof(x)=x$ for all $x\neq\frac{2}{3}$. What is the inverse of f ?

10. Consider $f\to R^{+}\left[-5,\infty\right)$ given by $f(x)=9x^{2}+6x-5$. Show that f is invertible with $f^{-1}(x)=\frac{\sqrt{x+6}-1}{3}$

11. Let $A=R-\{3\}$ and $B=R-\{1\}$. Consider the function $f:A\to B$ defined by $f(x)=\frac{x-2}{x-3}$. Show that f is one-one and onto and hence fined f^{-1}.

12. Consider the function $f:R^{+}\to\left[-9,\infty\right)$ given by $f(x)=5x^{2}+6x-9$. Prove that f is invertible with $f^{-1}(y)=\frac{\sqrt{54+5y}-3}{5}$.

13. If $f(x)=x+7$ and $g(x)=x-7$, $x\in R$, write fog (7).

14. What is the range of the function $f(x)=\frac{|x-1|}{x-1}$?

15. If $R\to R$ be defined by $f(x)=(3-x^{3})^{1/3}$, then find $fof(x)$.

16. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and $f=\{(1,4),(2,5),(3,6)\}$ be a function from A to B. State whether f is one-one or not.

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Binary Operations

A binary operation * on a set x is a function $*: x \times x \to x$. we denote $*(a,b)$ by $a*b$, where $a,b \in x$.

Practice Problems

1. Let S = {1, 2, 3, 4} and * be an operation on S defined by a * b = r, where r is the least non – negative remainder when product is divided by 5. Prove that * is a binary operation on S.
2. On the set W of all non-negative integers * is defined by a * b = a^b. Prove that * is not a binary operation on W.
3. Determine whether or not each of the definition of * given below gives a binary operation. In the event that * is not a binary operation give justification of this.

   On R, defined by a * b = ab^2
   Here, Z^+ denotes the set of all non-negative integers.
4. Let * be a binary operation on the set I of integers, defined by a * b = 2a + b – 3. Find the value of 3 * 4.
5. Is * defined on the set {1, 2, 3, 4, 5} by a * b = LCM of a and bb a binary operation ? Justify your answer.
6. The binary operation $*: R \times R \to R$ is defined as a * b = 2a + b. Find (2 * 3) * 4.
7. Let * be a binary operation on N given by a * b = LCM (a, b) for all $a,b \in N$. Find 5 * 7

Types Of Binary Operations

Commutativity

A binary operation '*' on a set S is said to be a commutative binary operation, if a * b = b * a for all $a,b \in S$

Associativity

A binary operation '*' on a set S is said to be an associative binary operation, if (a * b) * c = a * (b * c) for all $a,b \in S$.

Distributivity

Let S be a non-empty set and * and '$\odot$' be two binary operations on S. Then, '*' is said to be distributive over o, if for all $a,b,c \in S$

a * (b $\odot$ c) = (a * b) $\odot$ (a * c) and (b $\odot$ c) * a = (b * a) $\odot$ (c * a)

Practice Problems

1. Discuss the commutative and associativity of the binary operation '*' on R defined by a * b = a + b + ab for all $a,b \in R$, where on RHS we have usual addition, subtraction and multiplication of real numbers.
2. Discuss the commutative and associativity of binary operation '*' defined on Q by the rule a * b = a – b + ab for all $a,b \in Q$.
3. Let '*' be a binary operation on N given by a * b = HCF (a, b) for all $a,b \in N$.

   (i) Find : 12 * 4, 18 * 24, 7 * 5

   (ii) Check the commutativity and associativity of '*' on N.
4. Consider the binary operations $*R \to R$ and $0 : R \times R \to R$ defined as a * b = |a-b| and aob = a for all $a,b \in R$. Show that * is distributive over o. Dose o distributive over * ? Justify your answer.
5. Let A = N × N and '*' be a binary operation on A defined by (a, b) * (c, b) = (ac, bd) for all $a,b,c,d \in N$. Show that '*' is commutative and associative binary operation on A.

6. Let A be a set having more than one element. Let '*' be a binary operation on A defined by a * b = a for all $a, b \in A$. Is '*' commutative or associative on A ?
7. Determine which of the following binary operations are associative and which are commutative: *on Q defined by

   $a * b = \frac{a+b}{2}$ for all $a, b \in Q$
8. Let S be the set of all rational numbers except 1 and * be defined on S by a * b = a + b – ab, for all $a, b \in S$.

   Prove that : * is commutative as well as associtative.

Identity Element

Let '*' be a binary operation on a set S. If there exists an elements $e \in S$ such that a * e = a = e * a for all $a \in S$. Then e is called an identity elements for the binary operation '*' on set S.

1. If * is defined on set R of real numbers by $a * b = \frac{3ab}{7}$ find the identity elements in R for the binary operation *.
2. Find the identity elements in the set Q^+ of all positive rational numbers for the operation * defined by $a * b = \frac{ab}{2}$ for all $a, b \in Q^+$.
3. If * is defined on the set R of all real numbers by $a * b = \sqrt{a^2 + b^2}$ find the identity elements in R with respect to *.
4. If the binary operation * on the set Z is defined by $a * b = a + b - 5$ then find the identity element with respect to *.
5. On the set Z of integers, if the binary operation * is defined by $a * b = a + b + 2$ then find the identity element.

Inverse Of an Element

Let '*' be a binary operation on a set S, and let e be the identity element in S for the binary operation * on S. Then, an element $a \in S$ is called an invertible element if there exists an element $b \in S$ such that a * b = e = b * a.

The element b is called an inverse of element a.

Theorem 1

Let '*' be an association binary operation on a set S with the identity element e in S. Then, the inverse of an invertible element is unique.

Theorem 2

Let * be an associative binary operation on a set S and a be an invertible elements of S. Then, $(a^{-1})^{-1} = a$.

Practice Problems

1. On Q, the set of all rational numbers, a binary operation * is defined by $a * b = \frac{ab}{5}$ for all $a, b \in Q$. Find the identity element for * in Q. Also, prove that every non-zero element of Q is invertible.
2. On the set R – {-1} a binary operation * is defined by a * b = a + b + ab for all $a, b \in R - \{-1\}$. Prove that * is commutative as well as associative on R – R {–1}. Find the identity element and prove that every element of R – {–1} is invertible.
3. Let '*' be a binary operation on Q_0 (set of all non-zero rational numbers) defined by $a * b = \frac{ab}{4}$ for all $a, b \in Q_0$. Then, find the

   (i) Identity element in Q_0.

   (ii) Inverse of an elements in Q_0.

4. Let '*' be a binary operation on N given by a * b = L.C.M (a, b) for all
   (i) Find 5 * 7, 20 * 16
   (ii) Is * commutative?
   (iii) Is * associative?
   (iv) Find the identity element in N
   (v) Which elements of N are invertible? Find them.

5. Let X be a non-empty set and let * be a binary operation on P(X) (the power set of set X) defined by $A * B = A \cup B$ for all $A, B \in p(X)$. Prove that '*' is both commutative and associative on P(X). Find the identity element with respect to '*' on P(X). Also, show that $\varphi \in P(X)$ is the only invertible element of P (X).

6. Let X be a non-empty set and let '*' be a binary operation on P(X) (the power set of X) defined by $A * B = A \cap B$ for $A, B \in P(X)$.
   (i) Find the identity element with respect to * in P(X).
   (ii) Show that X is the only invertible element of P(X).

7. Let X be a non-empty set and let '*' be a binary operation on P(X) (the power set of set X) defined by $A * B = (A - B) \cup (B - A)$ for all $A, B \in P(X)$. Show that.
   (i) ϕ is the identity element for * on P(x).
   (ii) A is invertible for all $A \in P(X)$ and the inverse of A is A itself.

8. Let $A = N \cup \{0\} \times N \cup \{0\}$ and let '*' be a binary operation on A defined by $(a,b) * (c,d) = (a + c, b + d)$ for all $(a,b)(c,d) \in A$.
   (i) '*' is commutative on A.   (ii) '*' is association on A.

9. Show that the number of binary operations on {1, 2} having 1 as identity and having 2 as inverse of 2 is exactly one.
10. Determine the total number of binary operations on the set S = {1, 2} having 1 as the identity element.
11. Let * be a binary operation on Q_0 (set of non-zero rational numbers) defined by $a*b=\frac{3ab}{5}$ for all $a,b \in Q_0$ .Show that * is commutative as well as associative. Also, find its identity element, if it exists.
12. Let * be the binary operation on N defined by a * b = HCF of a and b. Does there exist identity for this binary operation on N?
13. Consider the set S = {1, 2, 3, 4}. Defined a binary operation * on S as follows: a * b = r, where r is the least non-negative remainder when ab is divined by 5.
14. Consider the infimum binary operation $\wedge$ on the set S = {1, 2, 3, 4, 5} defined by $a \wedge b$ = Minimum of a and b. write the composition table of the operation $\wedge$.
15. Consider a binary operation * on the set {1, 2, 3, 4, 5} given by the following multiplication table

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

(i) Compute (2 * 3) * 4 and 2 * (3 * 4)

(ii) Is * commutative ? (iii) Compute (2 * 3) * (4 * 5)

16. Define a binary operation * on the set A = {0, 1, 2, 3, 4, 5} as a * b = a + b (mod 6).

Or

A binary operation * on the set {0, 1, 2, 3, 4, 5} is defined as

$a*b=\begin{cases} a+b & if\ a+b<6 \\ a+b-6 & if\ \text{a+b} \geq 6 \end{cases}$. show that zero is the identity for this operation and each element 'a' of the set is invertible with 6 – a, being the inverse of 'a'.

17. Defined a binary operation * on the set A = {1, 2, 3, 4} as a * b = ab (mob5). Show that 1 is the identity for * and all element of the set A are invertible with $2^{-1}=3$ and $4^{-1}=4$.

18. Define a binary operation * on the set {0, 1, 2, 3, 4, 5} as

$a*b=\begin{cases} a+b & if\ a+b<6 \\ a+b-6 & if\ \text{a+b} \geq 6 \end{cases}$ Show that 0 is the identity for this operation and each element $a \neq 0$ of the set is invertible with 6 – a being the inverse of a.

19. Let * be a binary operation, on the set of all non-zero real numbers, given by $a*b=\frac{ab}{5}$ for all $a,b \in R-\{0\}$. Write the value of x given by 2 * (x * 5) = 10.

20. Let * be a binary operation defined by a * b = 3a + 4b – 2. Find 4 * 5.

21. If the binary operation * on the set Z of integers is defined by a * b = a + 3b^{2,} find the value of 2 * 4.

22. Let * be a binary operation on N given by a * b = HCF (a, b), $a,b \in N$. write the value of 22 * 4.

23. Let * be a binary operation on set of integers I, defined by a * b = 2a + b – 3. Find the value of 3 * 4.

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04 | Inverse Trigonometric Functions

Domain and Range of Inverse Trigonometric Function

The range of trigonometric function become the domain of inverse trigonometric function and restricted domain of trigonometric function becomes range or principal value branch.

Function	**Domain**	**Range** (Principal value branch)
$\sin^{-1} x$	[-1, 1]	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	[-1, 1]	$[0, \pi]$
$\tan^{-1} x$	R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	R	$(0, \pi)$
$\sec^{-1} x$	R- (-1,1)	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\csc^{-1} x$	R- (-1, 1)	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Practice Problems

1. Find the principle values of (i) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (ii) $\sin^{-1}\left(\frac{1}{2}\right)$

2. Find the domain of $f(x) = \sin^{-1} x + \cos x$.

3. If $x,y,z \in [-1, 1]$ such that $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = -\frac{3\pi}{2}$, find the value of $x^2 + y^2 + z^2$.

4. Find the principle values of (i) $\cos^{-1}\frac{\sqrt{3}}{2}$ (ii) $\cos^{-1}\left(-\frac{1}{2}\right)$

5. If $x,y,z \in [-1,1]$ such that $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$ then find the values of
   (i) $xy + yz + zx$
   (ii) $x(y + z) + y(z + x) + z(x + y)$

6. Find the principle values of each of the following :
   (i) $\tan^{-1}\left(-\sqrt{3}\right)$ (ii) $\tan^{-1}(1)$

7. For the principle values, evaluate each of the following :
   (i) $\tan^{-1}\left\{2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right\}$ (ii) $\cot^{-1}\left[\sin^{-1}\left\{\cos\left(\tan^{-1}1\right)\right\}\right]$

8. Which is greater, $\tan 1$ or $\tan^{-1} 1$?

9. Evaluate each of the following :
   (i) $\tan^{-1}1 + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

10. Find the set of values of $\sec^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

11. Find the principle values of $\sec^{-1}\frac{2}{\sqrt{3}}$ and $\sec^{-1}(-2)$

12. For the principle values, evaluate the following :

    $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$

13. Find the principle values of $\cot^{-1}\sqrt{3}$ and $\cot^{-1}(-1)$.

Properties Of Inverse Trigonometric Functions

In chapter 2, we have learnt that $f : A \rightarrow B$ is a bijection, then $f^{-1} : B \rightarrow A$ exists such than $f^{-1}$of (x) = x or, $f^{-1}(f(x)) = x$ for all $x \in A$.

1. $\sin^{-1}(\sin\theta) = \theta$ for all $\theta \in \left[-\pi/2,\ \pi/2\right]$
2. $\cos^{-1}(\cos\theta) = \theta$ for all $\theta \in [0,\ \pi]$
3. $\tan^{-1}(\tan\theta) = \theta$ for all $\theta \in \left[-\pi/2,\ \pi/2\right]$
4. $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta$ for all $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right) \cup \left(0, \frac{\pi}{2}\right]$
5. $\sec^{-1}(\sec\theta) = \theta$ for all $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
6. $\cot^{-1}(\cot\theta) = \theta$ for all $\theta \in (0, \pi)$

Practice Problems

1. Evaluate each of the following :

   (i) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ (ii) $\sin^{-1}\left(\sin(-600^0)\right)$

   (iii) $\cos^{-1}\left(\cos(-680^0)\right)$

2. Express each of the following in the simplest form :

(i) $\tan^{-1}\left\{\sqrt{\frac{1-\cos x}{1+\cos x}}\right\}$, $-\pi < x < \pi$

(ii) $\tan^{-1}\left\{\frac{\cos x}{1+\sin x}\right\}$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(iii) $\tan^{-1}\left\{\frac{\cos x}{1-\sin x}\right\}$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(iv) $\tan^{-1}\left\{\frac{\cos x-\sin x}{\cos x+\sin x}\right\}$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

3. Prove that : $\cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}=\frac{x}{2},\ 0<x<\frac{\pi}{2}$

4. Prove that :

$$tan^{-1}\left\{\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\}=\frac{\pi}{4}-\frac{x}{2},\ \textit{if}\ \pi < x<\frac{3\pi}{2}$$

5. Prove that :

$$\cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}=\frac{\pi}{2}-\frac{x}{2},\ \text{if}\ \frac{\pi}{2}<x<\pi$$

6. Write the following functions in the simplest form :

(i) $tan^{-1}\left\{\frac{x}{\sqrt{a^2-x^2}}\right\}, -a<x<a$ (ii) $\cos^{-1}\left\{\frac{x}{\sqrt{x^2+a^2}}\right\}$

7. $tan^{-1}\left\{\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right\}=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x,\quad 0 < x < 1$

8. $tan^{-1}\left\{\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right\}=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x^2,\quad -1 < x < 1$

9. $\cos^{-1}\left(\frac{3}{5}\cos x+\frac{4}{5}\sin x\right)$ where $-\frac{3\pi}{4}\le x\le\frac{\pi}{4}$

10. Simplify: $\sin^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right),\ \frac{\pi}{4}<x<\frac{\pi}{4}$

11. Evaluate the following: (i) $\cos^{-1}(\cos 10)$ (ii) $\tan^{-1}[\tan(-6)]$

12. Evaluate each the following:

(i) $\sin^{-1}\left(\sin\frac{\pi}{6}\right)$

(ii) $\cos^{-1}\left\{\cos\left(-\frac{\pi}{4}\right)\right\}$

(iii) $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

(iv) $\sec^{-1}\left(\sec\frac{\pi}{3}\right)$

(v) $\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{\pi}{4}\right)$

(vi) $\cot^{-1}\left(\cot\frac{\pi}{6}\right)$

(vii) $\tan^{-1}\left\{\sqrt{1+x^2}-x\right\},\ x\in R$

Property-II

If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ exists such that $fof^{-1}(x)$ or, $f(f^{-1}(x)) = x$ for all $x \in B$.

Property

(i)	$\sin(\sin^{-1} x) = x$	for all $x \in [-1,1]$
(ii)	$\cos(\cos^{-1} x) = x$	for all $x \in [-1,1]$
(iii)	$\tan(\tan^{-1} x) = x$	for all $x \in R$
(iv)	$\cos ec(\cos ec^{-1} x) = x$	for all $x \in (\infty,-1) \cup (1,\infty,)$
(v)	$\sec(\sec^{-1} x) = x$	for all $x \in (\infty,-1) \cup (1,\infty,)$
(vi)	$\cot(\cot^{-1} x) = x$	for all $x \in R$

ALGORITHM

Step I Obtain the expression and express it in the form $f(g^{-1}(x))$, where f and g are trigonometric functions.

Step II Express $g^{-1}(x)$ in terms of f^{-1} by using the following results:

$$\sin^{-1}\left(\frac{p}{h}\right) = \cos^{-1}\left(\frac{b}{h}\right) = \tan^{-1}\left(\frac{p}{h}\right) = \cos ec^{-1}\left(\frac{h}{p}\right) = \sec^{-1}\left(\frac{h}{b}\right) = ($$

where p, b and h denote respectively the perpendicular, base and hypotenuse of a right triangle.

Step III Let $g^{-1}(x) = f^{-1}(y)$. Replace $g^{-1}(x)$ by $f^{-1}(y)$ in $f(g^{-1}(x))$ and use property – II to get $f(g^{-1}(x)) = f(f^{-1}(y)) = y$.

1. Evaluate each of the following :

   (i) $\sin\left(\cos^{-1}\frac{4}{5}\right)$ (ii) $\sin\left(\sec^{-1}\frac{17}{15}\right)$

2. Evaluate each of the following :

   (i) $\cos\left(\cos^{-1}\frac{5}{13}\right)$ (ii) $\cos\left(\cos ec^{-1}\frac{13}{12}\right)$

3. Evaluate each of the following :

   (i) $\tan\left(\tan^{-1}\frac{3}{4}\right)$ (ii) $\tan\left(\cos^{-1}\frac{8}{17}\right)$

4. Evaluate : $\cos\left(\tan^{-1}x\right)$

5. Evaluate : $\cos\left(\sin^{-1}\frac{1}{4}+\sec^{-1}\frac{4}{3}\right)$

6. Find the value of the expression $\sin\left[\cot^{-1}\left\{\cos(\tan^{-1})\right\}\right]$

7. Prove that : $\tan^2(\sec^{-1}2)+\cot^2(\cos ec^{-1}3)=11$

8. Prove that : $\cos\left[\tan^{-1}\left\{\sin(\cot^{-1}x)\right\}\right]=\sqrt{\frac{x^2+1}{x^2+2}}$

9. If $\sin\left\{\cot^{-1}(x+1)\right\}=\cos(\tan^{-1}x)$ them find x.

10. Solve the following equation for $\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\frac{3}{4}\right)$

11. If $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}=\alpha$ them prove that $x^2=\sin 2\alpha$

12. Prove that : $\tan\left\{\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right\}+\tan\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right\}=\frac{2b}{a}$

13. Prove that : $\cos\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)=\frac{6}{5\sqrt{13}}$

Properties III

(i) $\sin^{-1}(-x)=-\sin^{-1}x$ for all $x\in[-1,1]$

(ii) $\cos^{-1}(-x)=\pi-\cos^{-1}x$ for all $x\in[-1,1]$

(iii) $\tan^{-1}(-x)=-\tan^{-1}x$ for all $x\in R$

(iv) $\operatorname{cosec}^{-1}(-x)=-\operatorname{cosec}^{-1}x$ for all $x\in(-\infty,-1)\cup(1,\infty)$

(v) $\sec^{-1}(-x)=\pi-\sec^{-1}x$ for all $x\in(-\infty,-1)\cup(1,\infty)$

(vi) $\cot^{-1}(-x)=\pi-\cot^{-1}x$ for all $x\in R$

Properties IV

(i) $\sin^{-1}\left(\frac{1}{x}\right)=\operatorname{cosec}^{-1}x$

(ii) $\cos^{-1}\left(\frac{1}{x}\right)=\sec^{-1}x$

(iii) $\tan^{-1}\left(\frac{1}{x}\right)=\begin{cases}\cot^{-1} & \textit{for } x>0\\ -\pi+\cot^{-1}x & \textit{for } x<0\end{cases}$

15. Evaluate : (i) $\cos\left\{\sin^{-1}\left(-\frac{5}{13}\right)\right\}$ (ii) $\cot\left\{\sin^{-1}\left(-\frac{7}{25}\right)\right\}$

14. Evaluate :(i) $\operatorname{cosec}\left\{\cos^{-1}\left(-\frac{12}{13}\right)\right\}$ (ii) $\tan\left\{\cos^{-1}\left(-\frac{12}{13}\right)\right\}$

15. Evaluate : $\operatorname{cosec}\left\{\cot^{-1}\left(-\frac{4}{3}\right)\right\}$

16. Prove that : $\sin^{-1}\left(-\frac{4}{5}\right)=\tan^{-1}\left(-\frac{4}{3}\right)=\cos^{-1}\left(-\frac{3}{5}\right)-\pi$

17. Prove that : $\tan^{-1}x+\tan^{-1}\frac{1}{x}=\begin{cases}\pi/2 & \textit{for } x>0\\ -\pi/2 & \textit{for } x<0\end{cases}$

Property - V

1. Find the value of $\cot(\tan^{-1}a+\cot^{-1}a)$.

2. If $-1\le x,\ y\le 1$ such that $\sin^{-1}x+\sin^{-1}y=\frac{\pi}{2}$, find the value of $\cos^{-1}x+\cos^{-1}y$.

3. If $\tan^{-1}x+\tan^{-1}y=\frac{4\pi}{5}$, find $\cot^{-1}x+\cot^{-1}y$.

4. If $\tan^{-1}x-\cot^{-1}x=\tan^{-1}\frac{1}{\sqrt{3}}$, find the value of x.

5. If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then find the value of x.

6. If $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$ then find x.

7. Prove that $\tan(\cot^{-1}x)=\cot(\tan^{-1}x)$. State with the reason weather the equality is valid for all value of x.

8. Find the greatest and least values of $(\sin^{-1} x)^2 + (\cos^{-1} x)^2$
9. Find the maximum and minimum values of $(\sin^{-1} x)^3 + (\cos^{-1} x)^3$,where $-1 \le x \le 1$.
10. Evaluate: $\cot(\tan^{-1} a + \cot^{-1} a)$
11. Solve: $\sin\left\{\sin^{-1}\frac{1}{5} + \cos^{-1} x\right\} = 1$.

Property – VI

(i)

$$\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right) & \text{if } xy < 1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) & \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) & \text{if } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$

(ii)

$$\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\left(\frac{x-y}{1+xy}\right) & \text{if } xy < -1 \\ \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right) & \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right) & \text{if } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$

1. Prove that : $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi$

2. Prove that : $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}=\tan^{-1}\frac{2}{9}$

3. Prove that : $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

4. Prove that : $\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19}=\frac{\pi}{4}$

5. Prove that : $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

6. Prove that : $\cot^{-1}7+\cot^{-1}8+\cot^{-1}18+\cot^{-1}3$

7. Prove that : $\tan^{-1}\left(\frac{a\cos x-b\cos x}{b\cos x+a\sin x}\right),-\frac{\pi}{2}<x<\frac{\pi}{2},\frac{a}{b}\tan x>-1$

8. Prove that : $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right),-\frac{1}{\sqrt{3}}<x<\frac{x}{a}<\frac{1}{\sqrt{3}}$

9. Prove that : $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$

10. Prove that : $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$

11. Prove that : $\tan^{-1}\sqrt{x^2+x}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$

12. If a1, a2, a3, …………, an are in arithmetic progression with common difference d, then evaluate the following expression:

$$\tan\left\{\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\tan^{-1}\left(\frac{d}{1+a_3a_4}\right)+\ldots\ldots+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right\}$$

13. Find the value of $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$

14. solve the following equation for x:

$\tan^{-1}\left(\frac{1-x}{1+x}\right) - \frac{1}{2}\tan^{-1}$ *where* $x = 0x > 0$

15. solve the following equation for x :

$\tan^{-1}(x+2) + \tan^{-1}(x-2) = \tan^{-1}\left(\frac{8}{79}\right)$

16. solve the following equation for x:

$\tan^{-1}\frac{x}{2} + \tan^{-1}\frac{x}{3} = \frac{\pi}{4}, \quad 0 < x < \sqrt{6}$

17. solve the following equation for x:

$\tan^{-1}\left(\frac{x-2}{x-4}\right) + \tan^{-1}\left(\frac{x+2}{x+4}\right) = \frac{\pi}{4}$

18. solve the following equation for x:

$\tan^{-1}(2+x) + \tan^{-1}(2-x) = \tan^{-1}\frac{2}{3}$ where $x < -\sqrt{3}$ *or* $x > \sqrt{3}$

Property-VII

(i) $\sin^{-1}x + \sin^{-1}y =$

$$\begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right. & \text{if } -1 \le x,\ y \le 1 \text{ and } x^2 + y^2 \le 1 \text{ or if } xy < \\ \pi - \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right. & \text{if } 0 < x, y \le 1 \text{ and } x^2 + y^2 \\ -\pi - \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right. & \text{if } -1 \le 0, y < 0 \text{ and } x^2 + y \end{cases}$$

(ii) $\sin^{-1}x+\sin^{-1}y=$

$$\begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right. & \text{if } -1\le x,\ y\le 1 \text{ and } x^2+y^2\le 1 \text{ or if } xy< \\ \pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right. & \text{if } 0<x,y\le 1 \text{ and } x^2+y^2 \\ -\pi-\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right. & \text{if } -1\le x\le 0, 0<y<0 \text{ and } x^2+y \end{cases}$$

1. Prove that : $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{56}{65}$

2. Prove that : $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{77}{85}$

3. Prove that : $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}=\sin^{-1}\frac{16}{65}=\frac{\pi}{2}$

4. Prove that : $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\cos^{-1}\frac{36}{85}$

5. Prove that : $\sin^{-1}\frac{3}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{33}{65}=s\,in^{-1}\frac{56}{65}$

6. Prove that : $\sin^{-1}6x+\sin^{-1}6\sqrt{3}x=-\frac{\pi}{2}$

7. Prove that : $\sin^{-1}\left(\frac{63}{65}\right)=\sin^{-1}\left(\frac{5}{13}\right)+\cos^{-1}\left(\frac{3}{5}\right)$

8. Prove that : $\sin^{-1}\left(\frac{5}{13}\right)+\cos^{-1}\left(\frac{3}{5}\right)=\tan^{-1}\left(\frac{63}{16}\right)$

9. Prove that : $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)=\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

10. Solve : $\cos^{-1} x + \sin^{-1} x = \frac{x}{2} - \frac{\pi}{6}$

Property-VIII

(i) $\cos^{-1} x + \cos^{-1} y =$

$$\begin{cases} \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\} & if\ -1 \le x,\ y \le 1 \text{ and } x+y \ge 0 \\ 2\pi - \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\} & f\ -1 \le x,\ y \le 1 \text{ and } x+y \le 0 \end{cases}$$

(ii) $\cos^{-1} x - \cos^{-1} y$

$$\begin{cases} \cos^{-1}\left\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\right\} & if\ -1 \le x,\ y \le 1 \text{ and } x \le y \\ 2\pi - \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\} & f\ -1 \le y \le 0, 0 < x \le 1 \text{ and } x \ge y \end{cases}$$

1. Prove that : $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

2. Prove that : $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$

3. If $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \alpha$ Prove that: $\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha$

Property-IX

(i) $2\sin^{-1} x =$

$$\begin{cases} \sin^{-1}(2x\sqrt{1-x^2}) & if\ -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}, \\ \pi - \sin^{-1}(2x\sqrt{1-x^2}) & if\ \frac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin^{-1}(2x\sqrt{1-x^2}) & if\ -1 \le x \le -\frac{1}{\sqrt{2}} \end{cases}$$

(ii)

$$\begin{cases} \sin^{-1}(3x-4x^3) & if\ -\frac{1}{2} \le x \le \frac{1}{2}, \\ \pi - \sin^{-1}(3x-4x^3) & if\ \frac{1}{2} \le x \le 1 \\ -\pi - \sin^{-1}(3x-4x^3) & if\ -1 \le x \le -\frac{1}{2} \end{cases}$$

1. $\tan^{-1} x + \tan^{-1}\left\{\frac{2x}{1-x^2}\right\} = \tan^{-1}\left\{\frac{3x-x^3}{1-3x^2}\right\}, \ |x| < \frac{1}{\sqrt{3}}$

2. Evaluate : $\tan\left(2\tan^{-1}\frac{1}{5}\right)$

3. Prove that : $2\sin^{-1}\frac{3}{5} - \tan^{-1}\frac{17}{31} = \frac{\pi}{4}$

4. Prove that : $\tan\frac{1}{2}\left\{\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right\} = \frac{x+y}{1-xy}$ If $|x|<1, y>0$ and $xy<1$

5. show that : $2\tan^{-1}(-3) = -\frac{\pi}{2} + \tan^{-1}\left(-\frac{4}{3}\right)$

6. Simplify : $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right), -\frac{1}{\sqrt{3}} < \frac{x}{a} < \frac{1}{\sqrt{3}}$

7. Prove that : $2\tan^{-1}\frac{1}{5} + \sec^{-1}\frac{5\sqrt{2}}{7} + 2\tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

8. $\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3}$, -1<x<1

9. Show that : $2\tan^{-1}\left\{\tan\frac{\alpha}{2}\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right\}=\tan^{-1}\left(\frac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}\right)$

10. evaluate : $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)$

11. Evaluate : $\sin\left(2\tan^{-1}\frac{2}{3}\right)+\cos(\tan^{-1}\sqrt{3})$

12. Prove that : $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\cos^{-1}\frac{3}{5}=\frac{1}{2}\sin^{-1}\frac{4}{5}$

13. prove that $\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{3}=\frac{\pi}{4}$

14. Prove that : $2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\tan^{-1}\frac{4}{7}$

15. Prove that : $2\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{17}{31}=\frac{\pi}{4}$

16. Prove that : $2\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{17}{31}=\frac{\pi}{4}$

17. Prove that : $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{31}{17}$

18. Prove that : $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}=\frac{\pi}{4}$

19. Prove that : $\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{2\pi}{3}, x>0$

20. Prove that : $2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x), \quad x \neq \frac{\pi}{2}$

21. Prove that : $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4} \quad x + y + xy$

22. If $\tan^{-1}\left(\sqrt{3}\right) + \cot^{-1} x = \frac{\pi}{2}$ Then find x

23. What is the principle value of $\sin^{-1}\left(-\frac{1}{2}\right)$

24. What is the principle value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

25. Write the value of $\tan^{-1}\left\{2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right\}$

26. Write the value of $\tan^{-1}\left\{2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right\}$

27.Write the principle value of $\tan^{-1}\sqrt{3} + \cot^{-1}\sqrt{3}$

28. Write the principle value of $\cos^{-1}(\cos 680^{0})$

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05 | Algebra of Matrix

Matrix

Definition

A matrix is an ordered rectangular array of numbers or function. The number are called the element of matrix. It is denoted by the symbol [ ] or ( ).

i.e, $A = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Order Of Matrix

If a matrix has m rows and n colums , then its order is written as m×n. If a matrix has order m×n , then it has mn elements. i.e, $\left[a_{ij}\right]_{mxn}$

In above matrix have 3 rows and 3 columns, hence the order of matrix is 3×3.

Types Of Matrix

1. **Column Matrix:** A matrix which have only one column , is called a column matrix.

   e.g. $A = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$

2. **Row Matrix :** A matrix which have only one row , is called a row matrix.

   e.g. $A = \begin{bmatrix} 1 & 5 & 6 \end{bmatrix}$

3. **Zero or Null Matrix:** A matrix is said to be a null matrix , if its all elements are zero, and it is denoted by O.

e.g. $$o = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4. **Square Matrix:** A matrix which have equal no. of rows and columns , is called a square matrix.

e.g. $$B = \begin{bmatrix} 1 & 6 \\ 3 & 5 \end{bmatrix}$$

5. **Diagonal Matrix:** A square matrix whose all the elements except the diagonal element are zero, is called diagonal matrix.

e.g. $$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

6. **Scalar Matrix:** A diagonal matrix whose all diagonal elements are equal is called diagonal matrix.

e.g. $$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

7. Unit or Identity Matrix: A diagonal matrix whose all diagonal elements are '1' is called Identity matrix. It is denoted by I.

e.g. $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equality of Matrices:

Two matrix A and B are said to be equal , if

(i) Order of A and B are same.

(ii) Corresponding elements of A and B are same i.e, $a_{ij} = b_{ij}$

e.g. If $A = \begin{bmatrix} 1 & 6 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ a & b \end{bmatrix}$ are equal matrix then

x=1, y=6, a=3 b=5

Oprations On Matrix

Addition and Subtraction of Metrices

If $A = [a_{ij}]_{mxn}$ and $B = [b_{ij}]_{mxn}$ then , $A + B = [a_{ij} + b_{ij}]_{mxn}$

And $A - B = [a_{ij} - b_{ij}]_{mxn}$ $, 1 \le i \ge\le m,\ 1 \le j \le n$

Properties of Addition of matrices

(i) Commutative Law

If $A = [a_{ij}]_{mxn}$ and $B = [b_{ij}]_{mxn}$ are any two matricesthen A+B = B+A

(ii) Associative Law

For any three matrices $A = [a_{ij}]_{mxn}$, $B = [b_{ij}]_{mxn}$ and $C = [c_{ij}]_{mxn}$ Then , A + (B+C) = (A+B) +C

(iii) Existence of additive Identity

Let $A = [a_{ij}]_{mxn}$ then A+O = O+A = A.

(iv) Existence of additive Inverse

Let $A = [a_{ij}]_{mxn}$ then we have another matrix $-A = [a_{ij}]_{mxn}$ such that A+(-A) = (-A)+A = O. So matrix (-A) is called additive inverse. Of A.

Multiplication of a Matrix by Scalar

Let $A=\left[a_{ij}\right]_{mxn}$ be a matrix and k is scalar then $kA=\left[ka_{ij}\right]_{mxn}$

e.g. $k\begin{bmatrix}1 & 6\\3 & 5\end{bmatrix}=\begin{bmatrix}k1 & 6k\\3k & 5k\end{bmatrix}$

Properties of scalarMultiplication

For any two matrices $A=\left[a_{ij}\right]_{mxn}, B=\left[b_{ij}\right]_{mxn}$ then

(i) $k(A+B)=kA+kB$ where k is scalar

(ii) $(k+I)A=kA+IA$ where k and l is scalar

Properties of Multiplication of matrix

(i) Non Commutativity

If AB and BA are both defined , then it is not necessary that AB =BA.

(ii) Associative Law

For three matrices A, B and C then (AB)C = A(BC)

(iii) Multiplicative Identity For any square matrix A, then A. I=A

(iv) Distributive Law

For three matrices A, B and C

(a) A(B+C) = AB + AC (b) (A+B)C = AC + BC

Practice Problems

If a matrix has 8 elements, what are the possible orders it can have? What if it has 5 elements?

1) Construct a 2×2 matrix $A=\left[a_{ij}\right]_{mxn}$ whose elements a_{ij} are given by:

(i) $\frac{(i-j)^2}{2}$ (ii) $a_{ij}\frac{(i-j)^2}{2}$ (iii) $a_{ij}\frac{(i-2j)^2}{2}$

(iv) $a_{ij}\frac{(2i+j)^2}{2}$ (v) $a_{ij}\frac{|2i-3j|}{2}$ (vi) $a_{ij}\frac{|-3i+j|}{2}$

(vii) $a_{ij}=e^{2ix}$ sin xj

2) Construct a 3 x 4 matrix A= [a_{ij}] whose elements a_{ij} are given by: $a_{ij}=\frac{1}{2}|-3i+j|$

3) Find the value of a, b , c and d from the following equations:

$$\begin{bmatrix} 2a+b & 2\text{-}2b \\ 5c-d & 4c+3d \end{bmatrix}=\begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$$

4) If $\begin{bmatrix} x-y & z \\ 2x-y & \omega \end{bmatrix}=\begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$ find x, y, z, ω .

5) $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ 4x+6 & a-1 & 0 \\ b-3 & 3b & z+2c \end{bmatrix}=\begin{bmatrix} 0 & 6 & 3y-2 \\ 2x & 3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$ Obtain the values of a, b, c, x, y and z.

6) If $\begin{bmatrix} 2x+1 & 5x \\ 0 & y^2+1 \end{bmatrix}=\begin{bmatrix} x+3 & 10 \\ 0 & 26 \end{bmatrix}$, find the value of (x + y).

7) If $\begin{bmatrix} xy & 4 \\ z+6 & x+y \end{bmatrix}=\begin{bmatrix} 8 & \omega \\ 0 & 6 \end{bmatrix}$ then find the values of x, y, z and ω .

8) Find the values of a and b if A = B, where

A= $\begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix}$ $B = \begin{bmatrix} 2a+2 & b^2+2 \\ 8 & b^2-56 \end{bmatrix}$

9) Simplify: $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ \text{-}\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

10) Find X and Y, if $x+y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $x-y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

11) Find X, y, z, t if $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

12) Find a matrix X such that 2A + B + X = O, where $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$

and, B= $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$

13) Find X, Y if $y = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and 2x+y= $\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$

14) If $x-y = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $x+y = \begin{bmatrix} 3 & 5 & 1 \\ -1 & 1 & 4 \\ 11 & 8 & 0 \end{bmatrix}$ find X and Y.

15) If A= $\begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ and B= $\begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$ then find the matrix X of order 3 x2 such that 2A + 3X = 5B

16) Find x y z and t if $3\begin{bmatrix} x & 5 \\ z & y-3 \end{bmatrix}+\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 7 & 14 \\ 15 & 14 \end{bmatrix}$

17) If X and Y are 2x2 matrices , then solve following matrix equations for x and Y .

$$2x+3y=\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}, 3x+2y=\begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix}$$

Multiplication of matrices:

A= $[a_{ij}]_{mxn}$ and B= $[a_{jk}]_{nxp}$ are any two matricesthen product AB= $[C_{ik}]_{mxp}$ where $c_{ik}=\sum_{j=1}^{n} a_{ij}b_{jk}$.

i.e, If the ith row of A is $[a_{i1} \ a_{i2} \ a_{i3} \ldots\ldots\ldots a_{in}]$ and the kth column of

B is $\begin{bmatrix} b_{1k} \\ b_{2k} \\ : \\ b_{nk} \end{bmatrix}$ then.,

$$c_{ik}=a_{i1}b_{1k}+a_{i2}b_{2k}+a_{i3}b_{3k}+\ldots\ldots\ldots+a_{in}b_{nk}=\sum_{j=1}^{n} a_{ij}b_{jk} \ .$$

Practice Problems

1) If $A=\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B=\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A+B)^2=A^2+B^2$ find a and b .

2) Find the values of x such that $\begin{bmatrix} 1 & x & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix}=0$

3) If $A=\begin{bmatrix}1 & 0\\ -1 & 7\end{bmatrix}$ find k so that $A^2+8A+ki$

4) If $A=\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}$ and $B=\begin{bmatrix}0 & 4\\ -1 & 7\end{bmatrix}$, find $3A^2-2B+I$

5) If $A=\begin{bmatrix}\cos 2\theta & \sin 2\theta\\ -\sin 2\theta & \cos 2\theta\end{bmatrix}$, find A^2.

6) Let $A=\begin{bmatrix}2 & -1\\ 3 & 4\end{bmatrix}$, $B=\begin{bmatrix}5 & 2\\ 7 & 4\end{bmatrix}$, $C=\begin{bmatrix}2 & 5\\ 3 & 8\end{bmatrix}$.

Find a matrix D such that CD-AB = 0.

7) If $A=\begin{bmatrix}1 & 0\\ -1 & 7\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\ 0 & 12\end{bmatrix}$ then find' k so that $A^2=8A+KI$

8) If $\begin{bmatrix}2 & -1\\ 1 & 0\\ -3 & 4\end{bmatrix}A=\begin{bmatrix}-1 & -8 & -10\\ 1 & -2 & -5\\ 9 & 22 & 15\end{bmatrix}$ find A.

9) $\begin{bmatrix}x & -5 & -1\end{bmatrix}\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}\begin{bmatrix}x\\ 4\\ 1\end{bmatrix}=0$ find x.

10) If $A=\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}$, find A2 – 5A + 4I and hence find a matrix X such that $A^2-5A+4I+X=O$

11) Let $x^2 - 5x + 6$. Find f (A), if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

12) If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and f(x) =$x^2 - 2x - 3$, show that $f(A) = 0$

13) Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ and f(x)=$x^2 - 4x + 7$ Show that $f(A) = O$

Use this result to find A^5.

14) Let $f(x) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Show that

$f(x)f(y) = f(x+y)$.

15) Let $A = \begin{bmatrix} 0 & -\tan(\alpha/2) \\ \tan(\alpha/2 & 0 \end{bmatrix}$ and I be the identity matrix

of order 2. Show that I + A = (I - A) $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

16) If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, then show that A is a root of the polynomial

$f(x) = x^3 - 6x^2 + 7x + 2$.

17) If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, then prove that $A^2 - 4A - 5I = 0$.

On Principle Of Mathemtical Induction

Statement: Let P(n) be a statement involving positive integer n such that:

(i) P(1) is true I,e. the statement is true for n=1, and

(ii) P(m+1) is true whenever P(m) is true

Then P(n) is true for all positive integer n.

Practice Problems

18) Prove the following by the principle of mathematical induction:

If $A=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$, then $A^n=\begin{bmatrix}1+2n & -4n\\ n & 1-2n\end{bmatrix}$ for every positive integer n..

19) If $A_a=\begin{bmatrix}\cos\alpha & sin\beta\\ -\sin\alpha & \cos\alpha\end{bmatrix}$, then prove that (i) $A_\alpha A_\beta = A_{\alpha+\beta}$

(ii) $(A_\alpha)^n=\begin{bmatrix}\cos\alpha & sin\beta\\ -\sin\alpha & \cos\alpha\end{bmatrix}$ for every positive integer n.

20) If $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$, then prove that

$A^n=\begin{bmatrix}3^{n-1} & 3^{n-1} & 3^{n-1}\\ 3^{n-1} & 3^{n-1} & 3^{n-1}\\ 3^{n-1} & 3^{n-1} & 3^{n-1}\end{bmatrix}$ $n \in N$ for every positive integer n.

21) If $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$. Prove that $(aI+bA)^n = a^nI + na^{n-1}bA$

Where I is unit matrix of order 2 and n is a positive integer.

22) Let A, B be two matrices such that they commute, Show that for any positive integer n

(i) $AB^n = B^n A$ (ii) $(AB)^n = A^n B^n$

23) If $A = \begin{bmatrix} \cos\theta & i\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then prove by principle of mathematical induction that $A^n = \begin{bmatrix} \cos n\theta & i\sin n\theta \\ i\sin n\theta & \cos n\theta \end{bmatrix}$, for all $n \in N$.

Application Of Matrices

24) A trust fund has Rs. 30000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide Rs. 30000 among the two types of bonds. If the trust fund must obtain an annual total interest of (i) Rs. 1800 (ii) Rs. 2000.

25) To promote making of toilets for women, an organization tried to generate awarness through (i) house calls (ii) letters, and (iii) announcements. The cost for each mode per attempt is given blow:

(i)Rs. 50 (ii) Rs. 20 (iii) Rs. 40

The numbers of attempts made in three villages X, Y and Z are given below:

	(i)	(ii)	(iii)
X	400	300	100
Y	300	250	75
z	5000	400	150

Find the cost incurred by the organization for three villages separately, musing matrices.

26) There are 2families A and B. there are 4 men, 6 women and 2 children in family A, and 2 for man 2 women, and 4 children in family B. the recommend daily amount of calories is 2400 for man 1900 for women, 1800 for children and 45 grams of proteins for men 55 grams for omen and 33 grams foe children. Represent the above information using matrix. Using matrix multiplication, calculate the total requirement of calories and proteins for each of the two families. What awareness can you create among people about the planned diet from this question?

27) In a parliament election, a political party hired a public relation firm to promote its candidates in three ways – telephone, house calls and letters. The cost per contact (in paisa) is given in matrix A as

$$A = \begin{bmatrix} 140 \\ 200 \\ 150 \end{bmatrix} \begin{matrix} Telephone \\ Hou\sec alls \\ Letters \end{matrix}$$

The number of contacts of each type made in two cities X and Y is given in the matrix B as

$$A = \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{bmatrix} \begin{matrix} City\ x \\ City\ Y \end{matrix}$$

Find the total amount spent by the party in the two cities. What should one consider before casting his/her vote – party's promotional activity or their social activities?

3) Three schools A, B,C organized a mela for collecting funds for helping the rehabilitation of flood victims . they sold handmade fans, mats, and plates from recycled material at a cost of rs. 25, Rs. 100 and Rs. 50 each. The number of articles sold are given below:

Articles / School	A	B	C
Hand fans	40	25	35
Mats	50	40	50
Plates	20	30	40

Find the funds collected by each school separately by sailing the above articles . Also , find the total funds collected for the purpose. Find the funds collected by each school separately by sailing the above articles . Also , find the total funds collected for the purpose.

Transpose Of Matrix

Definition: Let A= [a_{ij}] be m×n matrix . then the transpose matrix of A is denoted by A^T or A'.

i.e, $(A^T)_{ij} = a_{ij}$ for all I = 1,2,3….. m ; j = 1,2,3….n.

e.g. If $A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 5 & 6 \\ 5 & 1 & 4 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 2 & 1 \\ 5 & 5 & 1 \\ 3 & 6 & 4 \end{bmatrix}$

Properties of transpose

(i) For any matrix A , $(A^T)^T$ = A

(ii) For any two matrix A and B of the same order, $(A+B)^T = A^T + B$.

(iii) If A is a matrix and k is scalar then $(KA)^T = KA^T$

(iv) For any two matrix A and B of the same order,.$(AB)^T = B^T A^T$

Practice problems

1) If $A=\begin{bmatrix}-1\\2\\3\end{bmatrix}$ and $B=\begin{bmatrix}-2 & -1 & -4\end{bmatrix}$, verify that $(AB)^T = B^T A^T$.

2) Find the value of x, y, z if the matrix $A=\begin{bmatrix}0 & 2y & z\\x & y & -z\\x & -y & z\end{bmatrix}$ the

$'A = I_3$

3) If $A=\begin{bmatrix}3\\2\\3\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0 & 4\end{bmatrix}$, verify that $(AB)^T = B^T A^T$.

4) If $A^T=\begin{bmatrix}3 & 4\\-1 & 2\\0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 2 & 1\\1 & 2 & 3\end{bmatrix}$ find $A^T - B^T$.

5) If $A=\begin{bmatrix}\cos\alpha & \sin\alpha\\-\sin\alpha & \cos\alpha\end{bmatrix}$ then verify that $A^T A = I_2$.

6) If $A=\begin{bmatrix}\sin\alpha & \cos\alpha\\-\cos\alpha & \sin\alpha\end{bmatrix}$, verify that $A^T A = I_2$.

Symmetric And Skew-Symmetric Matrices

Symmetric Matrix A square matrix A= [a_{ij}] is called a symmetric matrix , if $a_{ij} = a_{ji}$

i.e, $A^T = A$

e.g. If $A = \begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 2 \\ -5 & -2 & 0 \end{bmatrix}$ Then $A^T = \begin{bmatrix} 0 & 3 & -5 \\ -3 & 0 & -2 \\ 5 & 2 & 0 \end{bmatrix}$ is symmetric matrix.

Skew Symmetric A square matrix A= [a_{ij}] is called a skew symmetric matrix , if $a_{ij} = -a_{ji}$

i.e, $A^T = -A$

e.g. If $A = \begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 2 \\ -5 & -2 & 0 \end{bmatrix}$ Then $A^T = \begin{bmatrix} 0 & 3 & -5 \\ -3 & 0 & -2 \\ 5 & 2 & 0 \end{bmatrix}$

Thus $A^T = -A$ is skew symmetric matrix.

Practice problems

7) If the matrix $A = \begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$ is skew-symmetric, find the value of a, b and c.

8) Let A be a square matrix. Then,
   (i) $A + A^T$ is a symmetric matrix
   (ii) $A - A^T$ is a skew-symmetric matrix.

9) Prove that every square matrix can be uniquely expressed ads the sum of a symmetric matrix and a skew-symmetric matrix.

10) If A and B are symmetric matrices, then show that AB is symmetric if AB = BA i.e. A and B commute.

11) Show that the matrix B^TAB is symmetric or skew-symmetric according A is symmetric or skew-symmetric.

12) Show that matrix $B^T AB$ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.

13) Let A and B be symmetric matrices of the same order. Then sow that

    (i) AB –BA is a skew-symmetric matrix.

    (ii) AB +BA is a symmetric matrix.

14) If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, prove that $A\text{-}A^T$ is a skew-symmetric matrix.

15) Express the matrix $A \begin{bmatrix} 4 & 2 & -1 \\ 3 & 5 & 7 \\ 1 & -2 & 1 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

16) Express the matrix $A \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$ as the sum of a symmetric and skew-symmetric matrix and verify your result.

17) A matrix which is both symmetric as well as skew-symmetric is null matrix.

Singular Matrix

Definition: If |A| = **0**

1) For what value of x the matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{bmatrix}$ is singular?

2) Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent

of θ .

3) $A=\begin{bmatrix}1+x & 7\\ 3-x & 8\end{bmatrix}$ For what value of x the matrix A is singular ?

Adjoint And Inverse Of A Matrix

Adjoint Matrix

If A= $[a_{ij}]_{n\times n}$ is a matrix $[A_{ij}]^T_{n\times n}$ where A_{ij} is a cofactor of element .

Inverse Matrix

If A is a non singular matrix (I,e. $|A|\neq 0$), then

$$A^{-1}=\frac{1}{|A|}adj(A)$$

Practice Problems

1) If $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$, show that $A^2-5A+7I=O$. hence find A^{-1}.

2) If $A=\begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix}$, find the value of μ so that $A^2=\mu A-2I$ hence find A^{-1}

3) For the Matrix $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -1\\ 2 & -1 & 3\end{bmatrix}$. Show that $A^3-6A^2+5A+11I_3=O$ hence find A^{-1}.

4) If $A^{-1}=\begin{bmatrix}3 & -1 & 1\\ -15 & 6 & -5\\ 5 & -2 & 2\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}$ find $(AB)^{-1}$

5) If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$, find $(A^T)^{-1}$

6) Find the adjoint of the matrix $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ and hence show that A (adj A) = $I_{3.}$

Elementary Transformation Or Elementary Operatio

1) Use elementary column operations $C_2 \to C_2 \to 2C$ in the matrix equation $\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

2) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ 3 & 3 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ by using elementary row transformations.

3) Find the inverse of each of the following matrices by using elementary row transformations:

(i) $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & -1 & 4 \\ 4 & 0 & 2 \\ 3 & -2 & 7 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ (iv) $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

(v) $A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

4) If A is a square matrix of order 3 such that |A| = 5, write the value of .

5) If C_{ij} is the cofactor of the element a_{ij} is the cofactor of the element a_{ij} of the matrix $A = \begin{bmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{bmatrix}$, then write the value of $a_{32}C_{32}$.

6) If $A = \begin{bmatrix} 3 & 1 \\ 2 & -3 \end{bmatrix}$, then find .

7) If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, write A^{-1} in terms of A.

Solution Of Simultaneous Liner Equations By Using Matrix Method

Let the given system of equation be $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$, we can write above equation in matrix form as AX = B where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \text{ hence , } x = A^{-1}B$$

where $A^{-1} = \frac{1}{|A|} adj(A)$ $|A| \neq 0$

Practice Problems

Solve the following system of equations, using matrix method:

1) x+y+z=7, x+3z=11, x-3y = 1 [ans. x=2, y=1, z=3]
2) X+y+z = 3 , 2x-y+z = -1, 2x+y-3z = -9
[x= -8/7, y = 10/7, z= 19/7]
3) 2/x -3/y +3/z = 10, 1/x+ 1/y+1/z = 10, 3/x -1/y + 2/z = 13
[x=1/2 , y= 1/3 , z= 1/5]
4) 2/x +3/y+10/z = 4, 4/x-6/y+5/z = 1, 6/x+9/y-20/z =2.
[X=3, y=2, z=1]
5) X-y+2z = 7, 3x +4y-5z = -5, 2x – y + 3Z =12
[X=2, Y=1, Z=3]
6) X-Y = 3, 2X+3Y+4Z = 17, Y+2Z = 7
[x=2, y=-1, z=4]
7) 2x-3y+5z=11, 3x+2y-4z = -5, x+y-2z = -3
[x=1, y=2, z=-1]
8) X+2y+5z = 10, x-y-z = -2 , 2x+3y-z = -11
[x=-1, y=-3, z=3]
9) 3x-4y+2z = -1, 2x+3y 7, x+z = 2
[x=3, y=2, z=-1]
10) X-2y= 10, 2x+y+3z = 8 , -2y + z = 7
[x=4, y=-3,z=1]
11) If $A = \begin{bmatrix} 8 & -4 & 1 \\ 10 & 0 & 6 \\ 8 & 1 & 6 \end{bmatrix}$, then find A^{-1} and hence solve the following equation 8x-4y =z = 5 , 10x +6z = 4, 8x +y+6z = 5/2

12) If $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ then find AB . use this to solve thje system of equation x-y+z = 3, 2x+3y+4z = 5, 7x+3y-3z = 7.

13) If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find and hence solve the system of liner equations: 2x – 3y + 5z = 11, 3x + 2y – 4y = - 5, x + y – 2z = - 3.

14) Find A^{-1}, if $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$. Hence, solve the following system of liner equations:
x + 2y + 5z = 10, x – y – z = - 2, 2x + 3y – z = - 11.

15) $A = \begin{bmatrix} 3 & -4 & 2 \\ 2 & 3 & 5 \\ 1 & 0 & 1 \end{bmatrix}$, find and hence solve the following system of equations: 3x – 4y + 2z = - 1, 2x + 3y + 5z = 7, x + z = 2.

16) A = , find AB. Hence, solve the system of equations:
x – 2y = 10, 2x + y + 3z = 8 and -2y + z = 7.

17) If A = , find . Using **,** solve the system of liner equations.
x – 2y = 10, 2x – y – z = 8, - 2y + z = 7.

18) Given A = , find BA and use this to solve the system of equations y + 2z = 7, x – y = 3, 2x + 3y + 4z = 17.

PROBLEMS ON APPLICATION OF MATRICES

19) The management committee of a residential colony decided to award some of its members (say x) for honesty, some (say y) for helping others (say z) for supervising the workers to keep the colony neat and clean. The sum of all the awardees is 12. Three times the sum of awardees for cooperation and supervision added to two times the number of awardees for honesty is 33. If the sum of the number of awardees for honesty and supervision is twice the number of awardees for helping others, using matrix method, find the number of awardees of each category. Apart from these value, namely, honesty, cooperation and supervision, suggest one more value which the management must include for awards.

20) A school wants to award its students for the value of Honesty, Regularity and Hard work with a total each award of Rs. 6,000. Three times the award money for Hard work added to that given for honesty amounts to Rs. 11,000. The award money given for Honesty and Hard work together is double the one given for Regularity. Represent the above situation algebraically and find the award for each value, using matrix method. Apart from these values, namely, Honesty, Regularity and Hard work, suggest one more value which the school must include for awards.

21) Two institution decided to award their employees for the three values of resourcefulness, competence and determination in the form of prizes at the rate of Rs. x, Rs. y and Rs .z respectively per persons . the first institution decided to award respectively 4, 3 and 2 employees with a total prize money of Rs. 37000 and the second institution decide to award respectively 5, 3and 4 employees with a total prize money of Rs. 47000. If all the three prizes per person together amount to Rs. 12000, then using matrix method find the value of x, y and z. What values are described in this equations?

22) Two factories decided to award their employees for three values of (a0 adaptable to new techniques , (b) careful and alert in difficult situation and (c) keeping calm in tense situation , at the rate of Rs. x , Rs. y , and Rs. z per person respectively. The first factory decided to honor respectively 2, 4 and 3 employees with a total prize money of Rs. 29000. The second factory decided to honor respectively 5, 2 and 3 employees with the prize money of Rs. 30500. If the three prizes per persons together cost Rs. 9500, then solve these equations using matrices.

23) Two Schools A and b want to award their selected students on the values of sincerity , truthfulness and helpfulness. The school A wants to award Rs. x each, Rs. y each, Rs. z each for the three respectively values to 3, 2 and 1 student respectively with a total award money of Rs. 1600. School B wants to spend Rs. 2300 to awards its 4, 1 and 3 students on the respective values if the total amount of award for one prize on each value is Rs. 900, using matrices, find the award money for each value . Apart from these three values, suggest one more value which should be considered for award.

24) Two schools P and Q want to award their selected students on the values of discipline, politeness and punctuality. The school P wants to award Rs. x each, Rs. y each and Rs. z each for the three respectively values to its 3, 2 and 1 students with a total award money of Rs. 1000. School Q wants to spend rs. 1500 to award its 4, 1 and 3 students on the respective values. if the total amount of award for one prize on each value is Rs. 600, using matrices, find the award money for each value for award.

25) Two schools P and Q want to award their selected students on the values of Tolerence, kindness and Leadership. The school P wants to award Rs. x each, Rs. y each, and Rs. z each for the three respectively values to its 3, 2 and 1 students with a total award money of Rs. 2200. School Q wants to spend Rs. 3100

to award its 4, 1 and 3 students on the respective values. if the total amount of award for one prize on each value is Rs. 1200, using matrices , find the award money for each value for award.

26) A total amount of Rs. 7000 is deposited in three different saving bank accounts with annual interest rates 5% , 8% and 8 % respectively . The total annual interest from these three accounts is Rs. 550. Equal amounts have been deposited in the 55, and 8% .saving accounts. Find the amount deposited in each of the three accounts, with the help of matrices.

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06 | Determinants

Defination

Every square matrix can be associated to an expression or a number which is known as its determinant. *If* $A=\left[a_{ij}\right]$ is a square matrix of order n, then the determinant of A is denoted by det A or,

$$|A| \text{ or. } \begin{vmatrix} a_{11} & a_{12}\cdots & a_{1j} & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2j} & a_{2n} \\ a_{i1} & a_{i2}\cdots & a_{ij} & a_{in} \\ a_{n1} & a_{n2}\cdots & a_{nj} & a_{nn} \end{vmatrix}$$

Singular Matrix

Defination

A square matrix is a singular matrix if its determinant is zero. Otherwise, it is a non–singular matrix.

Illustration 1

For what value of x the matrix $A=\begin{bmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{bmatrix}$ is singular?

Ans. x = -1.

Minors And Cofactors

Minor Let $A=\left[a_{ij}\right]$ be a square matrix of order n. The minor M_{ij} of a_{ij} in A is the determinant of the square sub-matrix of order (n-1) obtained by leaving ${}_i th$ row and j^{th} column of A.

For example, if $A = \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix}$, then

M_{11} = Minor of $a_{11} = 2$, M_{12} = Minor of $a_{12} = -3$,

M_{21} = Minor of $a_{21} = -7$, M_{22} = Minor of $a_{22} = 4$,

Cofactor

Let $A = \left[a_{ij}\right]$ be a square matrix of order n. The cofactor c_{ij} of a_{ij} in A is equal to $(-1)^{i+j}$ times the determinant of the sub-matrix of order (n-1) obtained by leaving $_i th$ row and j^{th}column of A. It follows from this definition that

c_{ij} = Cofactor of a_{ij} in A $= (-1)^{i+j}$ where M_{ij} is minor of a_{ij} in A.

Thus, we have $c_{ij} = \begin{cases} M_{ij}\ i+j \text{ is even} \\ M_{ij} i+j \text{ is odd} \end{cases}$

For example, if $A = \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix}$, then

$c_{11} = (-1)^{1+1} M_{11} = M_{11} = 2, C_{12} = (-1)^{1+2} M_{12} = -(-3) = 3,$

$c_{21} = (-1)^{2+1} M_{21} = M_{21} - (-7) = 7, \quad c_{22} = (-1)^{2+2} M_{22} = 4$

1) Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of . θ

2) $\begin{vmatrix} a+ib & c+id \\ c+id & a-ib \end{vmatrix}$

3) Evaluate : Ц $\begin{vmatrix} 0 & \sin a & -\cos a \\ -\sin\beta & \cos\beta & \sin\beta \\ \sin\beta & \sin\beta & 0 \end{vmatrix}$

4) Evaluate : Ц $\begin{vmatrix} \cos a\cos\beta & \cos a\sin\beta & -\sin a \\ -\sin\beta & \cos\beta & 0 \\ \sin a\cos\beta & \sin a\sin\beta & \cos a \end{vmatrix}$

5) Find the values of x, if

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ (ii) $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$

6) For what value of x the matrix A is singular?

(i) $A = \begin{bmatrix} 1+x & 7 \\ 3-x & 8 \end{bmatrix}$

Properties Of Determinants

We have defined the determinant of a square matrix of order 4 or less. Infact, these definitions are consequences of the general definition of the determinant of a square matrix of any order which needs so many advanced concepts. These concepts are beyond the scope of this book. Using the said definition and some other advanced concepts we can prove the following properties. But, the concepts used in the definition itself are very advanced. Therefore we mention these properties and verify them for a determinant of a square matrix of order 3.

Property 1

Let $A\left[a_{ij}\right]$ be a square matrix of order n, then the sum of the product of elements of any row (column) with their cofactors is always equal to|A| or, det (A).

i.e. $\sum_{j=1}^{n} a_{ij}c_{ij} = |A|$ and $\sum_{i=1}^{n} a_{ij}c_{ij} = |A|$

Property2

Let $A=\left[a_{ij}\right]$ be a square matrix of order n. then the sum of the product of elements of any row (column) the cofactors of the corresponding elements of some other row (column) is zero.

i.e. $\sum_{j=1}^{n} a_{ij}c_{kj}=0$ and $\sum_{i=1}^{n} a_{ij}c_{ik}=0$

Property 3

Let $A=\left[a_{ij}\right]$ be a square matrix of order n, then $|A|=\left|A^{T}\right|$

By the abuse of language this property is also stated as follows:
The value of determinant remains uncharged if its rows and columns are interchanged.

Property 4

Let $A=\left[a_{ij}\right]$ be a square matrix of order $n(\geq 2)$ and let B be a matrix obtained from A by interchanging any two rows (columns) of A, then = $|B|=-|A|$

Conventionally this property is also stated as

If any two rows (columns) of a determinant are interchanged, then value of the determinant changes by minus sign only.

Property5

if any two rows (columns) of a square matrix $A=\left[a_{ij}\right]$ of order $n(\geq 2)$ are identical, then its determinant is zero i.e. $|A|=0$.

If any two rows of columns of a determinant are identical, then its value is zero.

Property 6

Let $A=\left[a_{ij}\right]$ be a square matrix of order n, and let B be the matrix obtained from A by multiplying each element of a row (column) 0f A by a scalar k, then $|B|=k|A|$

Conventionally this property is also stated as

If each element of a row (column) of a determinant is multiplied by a constant k, then the value of the new determinant is k times the value of the original determinant.

Property 7

Let A be a square matrix and B be a matrix obtained from A by adding to a row (column) 0f A a scalar multiple of another row (column) of A, then $|B| = |A|$

If each element of a row (column) of a determinant is multiplied by the same constant and then added to the corresponding elements of some other row (column), then the value of the determinant remain same.

Property 8

Let A be a squarematrix of order $n(\geq 2)$ such that element in a row (column) of A is zero, then $|A| = 0$

If each element of a row (column) of a determinant is zero, then its value is zero.

Property 9

If $A = \left[a_{ij}\right]$ is a diagonal matrix of order $n(\geq 2)$, then

$|A| = a_{11} \times a_{22} \times a_{33} \times .....\times a_{nn}$

Property 10

If A and B are square matrices of the same order, then

$|AB| = |A||B|$

Evaluation of Determinants

Type I Determinants in which two rows (columns) became identical by applying the properties of determinants.

Example 1 show that $\begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix} = 0$

Example 2 Without expanding prove that $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$.

Example 3 Find the value of the determinant $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 5 & 8 \\ 6x & 9x & 12x \end{vmatrix}$.

Ans. $3x \times 0 = 0$

Example 4 Without expanding show that $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$.

Example 5 if a, b, c are in A.P., find the value of

$$\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix} = 0.$$

Ans.0

Type II Evaluating Determinants by using the the properties of determinants and proving identities

Example 1 if $\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix}$ and $\begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$, without expanding prove that.

Example 2 Without expanding or evaluating show that

$$\begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix} = 0$$

Example 3 Prove that : $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$

Example 4 Evaluate : $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Ans. $(b-a)(c-a)(c-b)\times 1 = (a-b)(b-c)(c-a)$

Example 5 Show that : $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x)$

6. Prove that : $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}$

$= (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$

7. Prove that : $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

8. Show that . $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$

$= (a-b)(b-c)(c-a)(ab+bc+ca)$

9. then prove that $x \neq y \neq z$ $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$ $xyz = -1$

10. For any scalar p prove that $= \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$

$= (1+pxyz)(x-y)(y-z)(z-x)$

11. Using properties of determinants, show that $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$

12. if $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$ using properties of determinants, find the value of $f(2x) - f(x)$

Ans. $= ax(2a+3x)$

13. Show that : $\begin{vmatrix} x & p & q \\ p & x & q \\ q & q & x \end{vmatrix} = (x-p)(x^2+px-2q^2)$.

14. Prove that : $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$

15. Show that : $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix} = 1$

16. Show that : $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$

17. Show that : $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

18. Prove that :

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc+bc+ca+a$$

19. Prove that : $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$

20. Show that : $\begin{vmatrix} (b+c)^2 & ba & ca \\ ab & (c+a)^2 & cb \\ ac & bc & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$

21. Show that : $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

22. If a,b,c are positive and unequal, show that the value of the

determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is always negative.

23. if $a+b+c \neq 0$ $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$ then prove that $a = b = c$

24. show that : $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(a+b+c)$

25. Solve : . $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$

26. Solve : $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

27. Prove that : $\begin{vmatrix} bc-a^2 & ca-^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ac-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

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07 Continuity

Continuity at a Point

A Function $f(x)$ is continuous at a point $x = a$, if

$(LHL)_{x=a} = (RHL)_{x=a} = f(a)$ or $\lim_{x \to a} f(x) = f(a)$

Where , $(LHL)_{x=a} \lim_{x \to a^-} f(x)$ and $RHL)_{x=a} = \lim_{x \to a^+} f(x)$

To evaluate LHL and RHL of a function $f(x)$ at $x = a$, put $x = a - h$ and $x = a - h$ respectively, where h'!0.

Discontinuity of a Function

A function $f(x)$ is said to be discontinuous at $x = a$, if it is not continuous at $x = a$, i.e,

(i) $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ does not exit.

(ii) $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

(iii) $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) \neq f(a)$ or $\lim_{x \to a} f(x) \neq f(a)$

Algebra of continuous function

Suppose f and g are two real function, continuous at real number a then ,

(i) $f + g$ is continuous at $x = a$

(ii) $f - g$ is continuous at $x = a$

(iii) $f.g$ is continuous at $x = a$

(iv) (f / g) is continuous at $x = a$

(v) kf is continuous at k is any constant

Composition of Two Continuous Function

Suppose f and g are two real valued function such that (fog) is defined at a . If g is continuous at a and f is continuous at g (a), then (fog) is continuous at a.

Practice Problem

1. Show that the function $f(x)$ given by $\begin{cases} x\sin\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$ is continuous at $x=0$

2. Show that the function $f(x)$ given by

$$f(x)=\begin{cases} \dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} & when\ x\neq 0 \\ 0 & when x=0 \end{cases}$$

is discontinuous at $x=0$.

3. Discuss the continuity of the function $f(x)$ at $x=1/2$, where

$$f(x)=\begin{cases} \frac{1}{2}-x\ ; & 0\leq x<1/2 \\ 1\ ; & x=1/2 \\ \frac{1}{3}-x\ ; & 1/2<x\leq 1 \end{cases}$$

4. Show that the function $f(x)$ given by $f(x)=2x-|x|$ is continuous at x = 0.

5. If the function $f(x)$ given by $f(x)=\begin{cases} 3ax+b & if\ \ x>1 \\ 11 & if\ x=1 \\ 5ax-2b & if\ \ x<1 \end{cases}$ is continuous at $x=1$, find the value of a and b.

6. Let $f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2} & , \ if \ \ x<0 \\ a & , \ \ f \ x=0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}-4}} & , \ \ f \ \ x>0\end{cases}$ Determine the value of a so that $f(x)$ is continuous at x = 0.

7. If $f(x)=\dfrac{\sqrt{2}\cos x-1}{\cot x-1}, x\neq\dfrac{\pi}{4}$ Find the value of f so that $f(x)$ becomes continuous at $x=\dfrac{\pi}{4}$.

8. Prove that the greatest integer function [x] is continuous at all point except at integer points.

9. Show that the function $f(x)=|\sin x+\cos x|$ is continuous at $x=\pi$.

10. Find the value of 'a' for which the function f defined by $f(x)=\begin{cases}a\sin\dfrac{\pi}{2}(x+1) & if \ \ x\leq 0 \\ \dfrac{\tan x-\sin x}{x^3} & \ \ x>0\end{cases}$ is continuous at $x=0$.

11. Discuss the continuity of $f(x)=\begin{cases}2x-1 & x<0 \\ 2x+1 & x\geq 0\end{cases}$ at $x=0$.

12. Determine the value of constant k so that the function $f(x)=\begin{cases}kx^2 & \text{if } x\leq 2 \\ 3 & if \ \ x>2\end{cases}$ continuous at $x=2$.

13. Determine the value of constant k so that the function $f(x)=\begin{cases}\dfrac{\sin 2x}{5x} & \text{if } x\neq 0\\ k & if\ x=0\end{cases}$ is continuous at $x=0$.

14. Find the values of a so that the function $f(x)=\begin{cases}ax+5 & \text{if } x\leq 2\\ x-1 & if\ x>2\end{cases}$ is continuous at $x=2$.

15. Find the value of k if $f(x)$ is continuous at $x=\pi/2$, where $f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x} & x\neq\pi/2\\ k & x=\pi/2\end{cases}$

16. Find the value of k for which $f(x)=\begin{cases}\dfrac{1-\cos 4x}{8x^2} & \text{when } x\neq 0\\ k & when\ x=0\end{cases}$ is continuous at $x=0$.

17. In each of the following, find the value of the constant k so that the given function is continuous at the indicate point :

i. $f(x)=\begin{cases}k(x^2-2x) & \text{if } x<0\\ \cos x & if\ x\geq 2\end{cases}$ at $x=0$.

ii. $f(x)=\begin{cases}kx+1 & \text{if } x\leq\pi\\ \cos x & if\ x>5\end{cases}$ at $x=\pi$.

iii. $f(x)=\begin{cases}\dfrac{x^2-25}{x-5} & x\neq 5\\ k & x=5\end{cases}$ at $x=5$.

iv. $f(x)=\begin{cases} kx^2 & \text{if } x\geq 1 \\ 4 & if\ x>1 \end{cases}$ at $x=1$.

v. $f(x)=\begin{cases} k(x^2+2) & \text{if } x\leq 0 \\ 3x+1 & if\ x>0 \end{cases}$ at $x=0$.

vi. $f(x)=\begin{cases} \dfrac{x^3+x^2-16x+20}{(x-2)^2} & x\neq 2 \\ k & x=2 \end{cases}$ at $x=0$.

18. $f(x)=\begin{cases} 1 & , & if\ x\leq 0 \\ ax+b & , & if\ 3<x<5 \\ 7 & , & if\ x\geq 5 \end{cases}$ is continuous at $x=3$ and $x=5$.

19. Discuss the continuity of the f (x) at the indicated points:

i. $f(x)=|X|+|X-1|$ at $x=0,1$.

ii. $f(x)=|x-1|+|x+1|$ at $x=-1,1$.

20. For what value of ë is the function $f(x)=\begin{cases} \lambda(x^2-2x) & \text{if } x\geq 0 \\ 4x+1 & if\ x\leq 0 \end{cases}$ continuous at $x=0$?

21. For what value of k is the following continuous at $x=2$?

$$f(x)=\begin{cases} 2x+1 & , & if\ x<2 \\ k & , & if\ x=2 \\ 3x-1 & , & if\ x>2 \end{cases}$$

22. Let $f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x} & , \; if \;\; x<\dfrac{\pi}{2} \\ a & , \; if \;\; x=\dfrac{\pi}{2} \\ \dfrac{b(1-\sin x)}{(\pi-2x)^2} & , \; if \;\; x=\dfrac{\pi}{2}\end{cases}$. If $f(x)$ is continuous

at $x=\dfrac{\pi}{2}$, find a and b.

23. If the function $f(x)$, decline below is continuous at $x=0$, find the value of k:

$$f(x)=\begin{cases}\dfrac{1-\cos 2x}{2x^2} & , \quad x<0 \\ k & , \quad x=0 \\ \dfrac{x}{|x|} & , \quad x>0\end{cases}$$

24. Find the relationship between 'a' and 'b' so that the function

f defined by $f(x)=\begin{cases} ax+1 & \text{if } x\le 3 \\ bx+3 & if \;\; x>3\end{cases}$ is continuous at $x=3$.

Continuity on interval

Continuity on an open interval

a function $f(x)$ is said to be continuous function an open interval (a,b) iff it is continuous at every point on the interval (a,b).

Continuity on a closed interval

a function $f(x)$ is said to be continuous function on a closed interval [a,b] if (i) it is continuous on the interval (a,b).

(ii) $\lim_{x\to a}+ f(x)=f(a)$ (iii) $\lim_{x\to b}- f(x)=f(b)$

Every where continuous function

A function $f(x)$ is said to be everywhere continuous function if it is continuous on the entire real line $(-\infty, \infty)$.

Practice Problems

25. Discuss the continuity of the function $f(x)$ given by

$$f(x)=\begin{cases} 2x-1 & \text{if } x<0 \\ 2x+1 & \textit{if } x\geq 0 \end{cases}.$$

26. Determine the value of the constant k so that the function

$f(x)=f(x)=\begin{cases} kx^2 & \text{if } x\leq 2 \\ 3 & \textit{if } x>2 \end{cases}$ is continuous.

27. $$f(x)=\begin{cases} |x+3| & , \text{ if } x\leq -3 \\ -2x & , \text{ if } -3<x<3 \\ 6x+2 & , \text{ if } x>3 \end{cases}$$

28. $$f(x)=\begin{cases} x^{10}-1 & \text{if } x\leq 1 \\ x^2 & \textit{if } x>1 \end{cases}$$

29. $$f(x)=\begin{cases} 2x & , \text{ if } x<0 \\ 0 & , \text{ if } 0\leq x\leq 1 \\ 4x & , \text{ if } x>1 \end{cases}$$

30. $$f(x)=\begin{cases} \sin x-\cos x & \text{if } x\neq 0 \\ -1 & \textit{if } x=0 \end{cases}$$

31. $$f(x)=\begin{cases} -2 & , \text{ if } x\leq -1 \\ 2x & , \text{ if } -1<x<1 \\ 2 & , \text{ if } x\geq 1 \end{cases}$$

32. $f(x)=\begin{cases} \dfrac{\sqrt{1+px}-\sqrt{1-px}}{x} & \text{if } -1\le x<0 \\ \dfrac{2x+1}{x-2} & \text{if } 0\le x\le 1 \end{cases}$

33. $f(x)=\begin{cases} 5 & , \text{ if } x\le 2 \\ ax+b & , \text{ if } 2<x<10 \\ 21 & , \text{ if } x\ge 10 \end{cases}$

34. $f(x)=\begin{cases} \dfrac{k\cos}{\pi-2x} & , \text{ if } x<\dfrac{\pi}{2} \\ 3 & , \text{ if } x=\dfrac{\pi}{2} \\ \dfrac{3\tan 2x}{2x-\pi} & , \text{ if } x>\dfrac{\pi}{2} \end{cases}$

35. Discuss the continuity of $f(x)=\sin|x|$.

36. Show that the function $g(x)=x-[x]$ is discontinuations at all integral points. Here $[x]$ denotes the greatest integer function.

37. Discuss the continuity of the following functions:

i. $f(x)=\sin x+\cos x$

ii. $f(x)=\sin x-\cos x$

iii. $f(x)=\sin x\cos x$

38. Show that $f(x)=\cos x^2$ is a continuous function.

39. Show that $f(x)=|\cos x|$ is a continuous function.

40. Find all the points of discontinuity of f defined by $f(x)=|x|-|x+1|$

41. Is $f(x) = f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ a continuous function ?

42. Given the function $f(x) = \dfrac{1}{x+2}$. Find the points of discontinuity of the function $f(f(x))$.

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08 Differentiation

Definition

The process of finding derivative of a function is called differentiation

Rules of Derivative

(i) Sum and Difference Rule Let $y = f(x) \pm g(x)$.

Then by using sum and difference rule, its derivative is written as $\frac{dy}{dx} = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$.

(ii) Product Rule Let $f(x)\ \ g(x)$. Then, by using product rule, it's derivative is written as

$$\frac{dy}{dx} = \left[\frac{d}{dx} f(x)\right] g(x) + \left[\frac{d}{dx} g(x)\right] f(x).$$

(iii) Quotient Rule Let $y = \frac{f(x)}{g(x)}$; $g(x) \neq 0$. then by using quotient rule, its derivative is written as

$$\frac{dy}{dx} = \frac{g(x) \times \frac{d}{dx}[f(x)] - f(x) \times \frac{d}{dx}[g(x)]}{\left[g(x)^2\right]}$$

(iv) Chain Rule Let $y = f(u)\, and\, u = f(x)$ then by using chain rule, we may write $\frac{dy}{du}\, and\, \frac{du}{dx}$ both exist.

Some standard Derivatives

(i) $\frac{d}{dx}(cons\tan t) = 0$ **(ii)** $\frac{d}{dx}(x^n) = nx^{n-1}$

(iii) $\frac{d}{dx}(e^x) = e^x$ **(iv)** $\frac{d}{dx}(\log_e x) = \frac{1}{x}, x > 0$

(v) $\frac{d}{dx}(a^x) = a^x \log_e a, \quad a > 0$ **(vi)** $\frac{d}{dx}(\sin x) = \cos x$

(vii) $\frac{d}{dx}(\cos x) = -\sin x$ **(viii)** $\frac{d}{dx}(\tan x) = \sec^2 x$

(ix) $\frac{d}{dx}(\cos ecx) = -\cos ecx \ \cot x$

(x) $\frac{d}{dx}(\sec x) = \sec x \ \tan x$ **(xi)** $\frac{d}{dx}(\cot x) = -\cos ec^2 x$

(xii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ **(xiii)** $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

(xiv) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ **(xv)** $\frac{d}{dx}(\cos ec^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

(xvi) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$ **(xvii)** $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

Practice Problem

Differentiation of a function of a function

1) Differentiate the following functions with respect to x:

(i) $\log \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$ Ans: =sec x.

(ii) $\log\left(x+\sqrt{a^2+x^2}\right)$ Ans: $=\frac{1}{\sqrt{a^2+x^2}}$

(iii) $\log\sqrt{\frac{1+\sin x}{1-\sin x}}$ Ans: sec x.

(iv) if $y=\left(x+\sqrt{x^2+a^2}\right)^n$ then prove that $\frac{dy}{dx}=\frac{ny}{\sqrt{x^2+a^2}}$

(v) if $y=\sqrt{\frac{1-x}{1+x}}$, prove that $(1-x^2)\frac{dy}{dx}+y=0$

(vi) if $(fx)=\sqrt{x^2+1}$ $g(x)=\frac{x+1}{x^2+1}$ and $h(x)=2x-3$

then find $f'(h'(g'(x))$

Differentiation of inverse trigonometric functions bychain rule
Differentiate the following functions with respect to x(1-57) :

(i) $\log\sqrt{\frac{1-\cos x}{1+\cos x}}$

(ii) y=log $\{\sqrt{x-1}-\sqrt{x+1}$ show that $\frac{dy}{dx}=\frac{-1}{2\sqrt{x^2-1}}$

(iii) if y= log $\sqrt{x+1}+\sqrt{x-1}$ prove that $\sqrt{x^2-1}\frac{dy}{dx}=\frac{1}{2}y$

(iv) if $y=\log\sqrt{\frac{1+\tan x}{1-\tan x}}$ prove that $\frac{dy}{dx}=\sec 2x$

(v) is y= (x-1)log (x-1)-(x+1)log (x+1), prove that

$$\frac{dy}{dx} = \log\left(\frac{x-1}{1+x}\right)$$

(vi) y=$\frac{1}{2}\log\left(\frac{1-\cos 2x}{1+\cos 2x}\right)$ prove that $\frac{dy}{dx} = 2\cos ec 2x$

(vii) $y = \sqrt{x^2 + a^2}$ prove that y$\frac{dy}{dx} - x = 0$

(vii) y= $\sqrt{a^2 - x^2}$ prove that y= $\frac{dy}{dx} + x = 0$

(ix)

$$\frac{d}{dx}\left\{\frac{x}{2}\sqrt{a^2} - x^2 + \frac{a^2}{2}\sin{-1}\frac{x}{a}\{\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin{-1}\frac{x}{a}\right\} = \sqrt{a^2 - x^2}$$

Dfferentiation by using trigonometrical substitution

(i) $1 + \cos 2x = 2\cos^2 x \; or, \cos 2x = 2\cos^2 x - 1$

(ii) $1 - \cos 2x = 2\sin^2 x \; or, \cos 2x = 1 - 2\sin^2 x$

(iii) cos2x=$\frac{1-\tan^2 x}{1+\tan^2 x}$

(iv) sin 3x = 3sinx-4sin$^3 x$

(v) $\tan 3x = \frac{3\tan x - \tan^{3x}}{1 - 3\tan^2 x}$

Following are some substitutions useful in finding derivatives:

Expression	**Substitution**
$a^2 + x^2$	x = a tan θ or, a cot θ
$a^2 - x^2$	x =a sinθ or,a cos θ
$x^2 - a^2$	x = a sec θ or, a cosec θ

$\sqrt{\frac{a-x}{a+x}}, or \sqrt{\frac{a+x}{a\_x}}$ x=a cos 20

$\sqrt{\frac{a^2-x^2}{a^2+x^2}} or, \sqrt{\frac{a^2+x^2}{a^2-x^2}}$ $x^2 = a^2 \cos 2\theta$

1. Differentiate $\tan-1\left(\frac{3x-x^3}{1-3x^2}\right)$ if $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

2. Differentiate each of the following functions with respect to x

(i) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x <$ (ii) $\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

(iii) $\cos^{-1}\left(\frac{2x}{1+x^2}\right), -1 < x < 1$ (iv) $\sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$

(v) $\tan^{-1}\left\{\sqrt{\frac{1-\cos x}{1+\cos x}}\right\}, -\pi < x < \pi$ (vi) $\tan^{-1}(\sec + \tan x), \frac{\pi}{2} < x < \frac{\pi}{2}$

(vii) $\tan^{-1}\left\{\frac{\sqrt{1+x^2-1}}{x}\right\}, x \neq 0$

(viii) $\tan^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}, 0 < x < \pi$

(ix) let $y = \tan^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}$

(x) $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right), -\frac{\pi}{2} < x < \frac{\pi}{2} and \frac{a}{b}\tan x > -1$

(xi) $\tan^{-1}\left(\dfrac{3a^2x-x^3}{a^3-3ax^2}\right), -\dfrac{1}{\sqrt{3}}<\dfrac{x}{a}<\dfrac{1}{\sqrt{3}}$

(xii) $\cos^{-1}\left\{\dfrac{\cos x+\sin x}{\sqrt{2}}\right\}, -\dfrac{\pi}{4}<x<\dfrac{\pi}{4}$

(xiii) $\sin^{-1}\left(\dfrac{2^{x+1}}{1+4^x}\right)$ (xiv) $\cot^{-1}\left(\dfrac{1-x}{1+x}\right)$

(i) if $y=\cot^{-1}\left\{\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}$ show that $\dfrac{dy}{dx}$ is independent of x

(ii) if $y=\tan^{-1}\left\{\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right\}$ find $\dfrac{dy}{dx}$

(iii) $y=\cos^{-1}\left\{\dfrac{2x-3\sqrt{1-x^2}}{\sqrt{13}}\right\}$ find $\dfrac{dy}{dx}$

(iv) differrentiate $\sin^{-1}\left\{\dfrac{2x^{x+1}.3^x}{1+(36)^x}\right\}$ with respect to x

Differentiation of implicittions

(i) if $x\sqrt{1+y}+y\sqrt{1+x}=0$ and $x \neq y$ prove that $\dfrac{dy}{dx}=-\dfrac{1}{(x+1)^2}$

(ii) if sin y= x sin (a+y) prove that $\dfrac{dy}{dx}=\dfrac{\sin^2(a+y)}{\sin a}$

Find $\dfrac{dy}{dx}$ in each of the following (i) :

(i) $(x^2+y^2)^2=xy$

(ii) if $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$ prove that $\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$.

(iii) if x$\sqrt{1+y+y}+\sqrt{1+x}=0$ prove that $(1+x)^2\frac{dy}{dx}+1=0$

(iv) if x sin (a+y) +sin a cos (a+y)=0 prove that $\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$

(v) if $e^x+e^y=e^{x+y}$ prove that $\frac{dy}{dx}=-\frac{e^x(e^y-1)}{e^y(e^x-1)} or, \frac{dy}{dx}+e^{y-x}=0$

(vi) if $\cos y=x\cos(a+y)$ with $\cos a\neq\pm1$ prove that $\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$

Logarithmic Differentiaton

Let y= if (x)$^{g(x)}$ Taking logarithm of both the sides, we get

log y=g(x).log$\{f(x)\}$

Differentiating with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx}=g(x)\times\frac{1}{f(x)}\frac{dy}{dx}(f(x))+\log\{f(x)\}.\frac{d}{dx}(g(x))$$

$$\therefore\frac{dy}{dx}=y\left\{\frac{g(x)}{f(x)}.\frac{d}{dx}(f(x))+\log\{f(x)\}.\frac{d}{dx}(g(x))\right\}$$

Differentiating with respect to x, we get

$$\frac{dy}{dx}=[f(x)]^{g(x)}\left\{\frac{g(x)}{f(x)}.\frac{d}{dx}(f(x))+\log\{f(x)\}.\frac{d}{dx}(g(x))\right\}$$

Differentiate the following functions with respect to x :

(i) $x^{\sin x}$

(ii) If y= $(\sin x)^{\tan x}+(\cos x)^{\sec x}$ find $\frac{dy}{dx}$

(iii) $x^{\cot x}+\frac{2x^2-3}{x^2+x+2}$

(iv) $x^y=e^{x-y}$, prove that $\frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}$

(v) If $x^y+y^x=2$ find $\frac{dy}{dx}$

(vi) If $x^y=y$ find $\frac{dy}{dx}$

(vii) If $(\cos x)^y=(\sin y)^x$ find $\frac{dy}{dx}$

(viii) If $y=a^x+e^x+x^x+x^a$ find $\frac{dy}{dx}$ at x=a

(ix) If $x^m y^n=(x+y)^{m+n}$ prove that $\frac{dy}{dx}=\frac{y}{x}$

(x) $x^{\cos x}+\frac{x^2+1}{x^2-1}$

(xi) $(x\cos x)^x+(x\sin x)^{1/x}$

(xii) $\left(x\frac{1}{x}\right)^x+x^{(x+\frac{1}{x})}$

(xiii) $e^{\sin x}+(\tan x)^x$

(xiv) $(\cos)^x+(\sin x)^{1/x}$

(xv) $x^{x^2-3}+(x-3)^{x^2}$

(xvi) $y=x^{\sin x}+(\sin x)^x$

(xvii) $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

(xviii) $y = x^{\cos x} + (\sin x)^{\tan x}$ (ii) $y = x^x + (\sin x)^x$

(xix) $y = x^{\log x} + (\log x)^x$

(xx) if $y^x = e^{y-x}$ prove that $\dfrac{dy}{dx} = \dfrac{(1+\log y)^2}{\log y}$

(xxi) $e^x + e^y = e^{x+y}$ prove that $\dfrac{dy}{dx} + e^{y-x} = 0$

(xxii) if $x\sin(a+y) + \sin a\cos(a+y) = 0$ prove that $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$

(xxiii) if $y = (\sin x - \cos x)^{\sin x - \cos x}$, $\dfrac{\pi}{4} < x < \dfrac{3\pi}{4}$ find $\dfrac{dy}{dx}$

(xxiv) if $xy = e^{x-y}$ find $\dfrac{dy}{dx}$

(xxv) if $y^x + x^y + x^x = a$ find $\dfrac{dy}{dx}$

(xxvi) if $(\cos x)^y = (\cos y)^x$ find $\dfrac{dy}{dx}$

(xxvii) if $\cos y = x\cos(a+y)$ where $\cos a \neq \pm 1$, prove that

$$\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

Differentiation of infinites series

(i) if $y = (\cos x)^{(\cos x)^{(\cos x)\ldots\infty}}$ prove that $\dfrac{dy}{dx} = \dfrac{y^2 \tan x}{(1 - y\log(\cos x))}$

(ii) if $y = (\sqrt{x})^{(\sqrt{x})^{(\sqrt{x})\ldots\ldots\infty}}$ show that $\dfrac{dy}{dx} = \dfrac{y^2}{x(2 - y\log x)}$

(iii) if $y = \cfrac{\sin x}{1 + \cfrac{\cos x}{1 + \cfrac{\sin x}{1 + \cos x......\infty}}}$ prove that

$$\frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1 + 2y + \cos x - \sin x}$$

Differentiation of parametric functions

I.Find $\frac{fy}{dx}$ in each of the following:

(i) $x = a\left\{\cos t + \frac{1}{2}\log\tan^2\frac{t}{2}\right\} y = a\sin t$

(ii) $x = a(\theta - \sin\theta) and y = a(1 - \cos\theta)$

(iii) $x = a\sec^3\theta$ *and* $y = a\tan^3\theta$ find $\frac{dy}{dx} at\theta = \frac{\pi}{3}$

II.Find $\frac{fy}{dx}$ when

(i) $x = at^2$ and $y = 2at$

(ii) $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$

(iii) $x = ae^\theta(\sin\theta - \cos\theta)$, $y = ae^\theta(\sin\theta + \cos\theta)$

(iv) $x = b\sin^2\theta$ *and* $y = a\cos^2\theta$

(v) $x = a(1 - \cos\theta)$ *and* $y = a(\theta + \sin\theta)$ *at* $\theta = \frac{\pi}{2}$

(vi) $x = a(\cos\theta + \theta\sin\theta)$ *and* $y = a(\sin\theta - \theta\cos\theta)$

(vii) $x = b\sin^2\theta and y = a\cos^2\theta = e^\theta\left(\theta + \frac{1}{\theta}\right) and y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

(viii) $x = 2\cos\theta - \cos 2\theta \ and \ y = 2\sin\theta - \sin 2\theta$ prove that $\frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$

(ix) if $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$ find $\frac{dy}{dx}$

(x) if $x = a\left(\frac{1+t^2}{1-t^2}\right) \ and \ y = \frac{2t}{1-t^2}$ find $\frac{dy}{dx}$

(xi) if $x = 10(t - \sin t) \ t = 12(1 - cost)$ find $\frac{dy}{dx}$

(xii) if $x = a(\theta - \sin\theta) \ and \ y = a(1 + \cos\theta)$ find $\frac{dy}{dx}\theta = \frac{\pi}{3}$

(xiii) if $x = a\sin 2t(1+\cos 2t), \ \ y = b\cos 2t(1-\cos 2t)$ fshow that

$t = \frac{\pi}{4}\frac{dy}{dx} = \frac{b}{a}$

(xiv) if $x = \cos t(3 - 2\cos^2 t), \ \ y = \sin t(3 - 2\sin^2 t)$ find the value of $\frac{dy}{dx}t = \frac{\pi}{4}$

if $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$ differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respwct to $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$

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09 Higher Order Derivative

Definition and notations

If $y = f(x)$ then $\frac{dy}{dx}$ is derivative of first order w.r.t. x similarly

$\frac{d^2 y}{dx^2}$ is the derivative of second order w.r.t. x then the alternative

notation for $\frac{dy}{dx}, \frac{d^2 y}{dx^2}, \frac{d^3 y}{dx^3}, \ldots\ldots\ldots\ldots, \frac{d^n y}{dx^n}$ are

$Y1, Y2, Y3, \ldots\ldots\ldots\ldots\ldots\ldots, Yn$

$y', y'', y''', \ldots\ldots\ldots\ldots, y^{(n)}$

$Dy, D^2 y, D^3 y, \ldots\ldots\ldots\ldots, D^n y$

$f'(x), f''(x), f'''(x), \ldots\ldots\ldots\ldots f^n(x)$

Practice problems

on proving relation involving various order derivatives of cartesian function

1. if $y = \sin^{-1} x$ show that $\frac{d^2 y}{dx^2} = \frac{x}{(1 - x^2)^{3/2}}$

2. if $y = Ae^{mx} + Be^{nx}$ show that $\frac{d^2 y}{dx^2} - (m + n)\frac{dy}{dx} + mny = 0$

3. if $y = A\cos(\log x) + B\sin(\log x)$ show that $x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + y = 0$

4. if $y = e^y (x + 1) = 1$ show that $\frac{d^2 y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

5. if $y = e^{y/x}(ax+b) = x$ show that $x^3 \frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$

6. if $y = \log x\left\{x + \sqrt{x^2 + a^2}\right\}$ show that $(x \ \ + a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

7. if $y = \sin^{-1} x$ show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$

8. if $y = e^{m\sin^{-1}x}$ show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$

9. if $y = \left\{x + \sqrt{x^2+1}\right\}^m$ show that $(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0$

10. if $y = \sin^{-1} x / \sqrt{1-x^2}$ show that $(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$

11. if $x = \tan\left(\frac{1}{a}\log y\right)$ show that $(1+x^2)\frac{d^2y}{dx^2} - (2x-a)\frac{dy}{dx} = 0$

12. if $y = x^x$ show that $\frac{d^2y}{dx^2}\frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$

On finding derivative of a parametric function

1. find $\frac{d^2y}{dx^2}$ if $x = at^2, y = 2at$

2.find $\frac{d^2y}{dx^2}$ if $x = a\cos^3\theta, y = a\sin^3\theta$ find its value at $\theta = \pi/6$

3. $x = a\sin t, y = a\left(cast + \log\tan\frac{t}{2}\right)$ find $\frac{d^2y}{dx^2}$

4. if $x = a\cos\theta + b\sin\theta, y = a\sin\theta - b\cos\theta$ prove that

$$y^2 \frac{d^2 y}{dx^2} - x\frac{dy}{dx} + y = 0$$

5. if $x = \sin t$ and $y = \sin pt$,show that $(1-x^2)\frac{d^2 y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0$

On proving relations involving various order derivatives

1. If $(x-a) + (y-b)^2 = c$ prove that $\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2 y}{dx^2}}$ is a constant

independent of a and b

2. Find the second order derivative of following function :

(i) $e^x \sin 5x$ (ii) $e^{6x}\cos 3x$ (iii) $\sin(\log x)$

(iv) $\log(\sin x)$ (v) $\tan^{-1} x$

3. if $y = e^{-x}\cos x$ show that $\frac{d^2 y}{dx^2} = 2e^{-x}\sin x$

4. if $y = x + \tan x$ prove that $\cos^2 x\frac{d^2 y}{dx^2} - 2y\frac{dy}{dx} + 2x = 0$

5. if $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta + \theta\cos\theta)$ prove that

$\frac{d^2 x}{d\theta^2} - a(\cos\theta - \theta\sin\theta)$, $\frac{d^2 x}{d\theta^2} - a(\sin\theta + \theta\cos\theta)$ and

$$\frac{d^2 y}{dx^2} = \frac{\sec^3\theta}{a\theta}$$

6. $x = a(1-\cos\theta), y = a(\theta + \sin\theta)$ prove that $\frac{d^2 y}{dx^2} = -1/a, at\theta = \pi/2$

7. if $x = \cos\theta, y = a\sin^3\theta$ prove that

$$\frac{d^2y}{dx^2} + \left(\frac{d^2y}{dx^2}\right)^2 = 3\sin^2\theta\,(5\cos^2\theta\ -1)$$

8. if $y = 3\cos(\log x) + 4\sin(\log x)$ prove that $x^2y_2 + xy_1 + y = 0$

9. $x = \sin\left(\frac{1}{a}\log y\right)$ show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

10. $(\tan^{-1}x)^2$ show that $(1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2$

11. Find $\frac{d^2y}{dx^2}$ where $y = \log\left(\frac{x^2}{e^2}\right)$

12. if $y = e^x(\sin x + \cos x)$ show that $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$

13. if $y = e^{a\cos^{-1}x}$ show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

14. if $y = 3e^{2x} + 2e^{3x}$ show that $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

15. if $x = \cos + \left(\log\tan\frac{t}{2}\right), y = \sin t$ find $\frac{d^2y}{dt^2}$ *and* $\frac{d^2y}{dx^2}$ *at* t = π /4

16. if x = a(cos2t+2t sin 2t),y = a(sin2t-2tcos2t) prove that $\frac{d^2x}{dx^2}$

17. if $x\cos(a+y) = \cos y$ then prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$ hence

show that $\sin a\frac{d^2y}{dx^2} + \sin 2(a+y)\frac{dy}{dx} = 0$

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10 Derivative as a Rate Measurer

Rate of change of quantities

Let $y = f(x)$ be a function of x then $\frac{dy}{dx}$ represent the rate of change of y with respect to x. also $(\frac{dy}{dx})x = x0$ represent the rate of change of y w.r.t. x at $x = x_0$.

Rate of change of two variables

If $x = f(t)$ and $y = g(t)$, then $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}$ where $\frac{dx}{dt} \neq 0$

Marginal cost and marginal revenue

Marginal cost

Marginal cost represent the instantaneous rate of change of the total cost at any level of output. If C(x) represent the cost function for x units produced , then marginal cost (MC) is given by

$$MC = \frac{d}{dx}\left\{C(x)\right\}$$

Marginal Revenue

Marginal revenue represent the rate of changer of total revenue w.r.t the number of items sold at an instant .

If R(x) is the revenue function for x units sold , then marginal revenue(MR) is given by

$$MR = \frac{d}{dx}\left\{R(x)\right\}$$

Practices problems

1. Find the rate of change of the area of a circle with respect to its radius. How fast is the area changing with respect to the radius is 3 cm?
2. The total cost C (x) associated with the production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 30x + 5000$. Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output.
3. The total revenue received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. Find the marginal revenue with respect to the number of items sold at an instant.
4. A car start from a point P at time t = 0 second and stops at point Q. The distance x, in metres, covered by it, in t second is given by $x = t^2(2 - \frac{t}{3})$. Find the time taken by it to reach at Q and also find distance between P and Q.
5. If x and y are the sides of two square such that $y = x - x^2$ Find the change of the area of second square with respect to the area of the first square.
6. A swimming pool is to be drained for cleaning. If L represents the number of litres of water in the pool t seconds after the pool has been plugged off to drain and $L = 200(10 - t)^2$. How fast is the water running out at the end of 5 second ? What is the average rate at which the water flows out during the first 5 seconds?
7. Find the rate of change of the volume of a sphere with respect to its surface area when the radius is 2 cm.

8. The total cost $C(x)$ associated with the production of x units of an item is given by

   $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$. Find the marginal cost when 17 units are produced.

9. The total revenue received from the sale of x units of a product is given by $R(x) = 13x^2 + 26 + 1 + 15$. Find the marginal revenue when $x = 7$.

10. The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (Marginal revenue). If the total revenue (in rupees) received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, find the marginal revenue, when $x = 5$, and write which value does the question indicate.

11. If the area of circle increases at a uniform rate, then prove that the perimeter varies inversely as the radius.

12. If then area of circle increase at a uniform rate, then prove that the perimeter varies inversely as the radius.

13. The side of an equilateral triangle is increasing at the rate of 2 cm/sec. At what rate is its area increasing when the side of the triangle is 20 cm?

14. A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of volume at any instant is proportional to the surface. Prove that the radius is decreasing at a constant rate.

15. A stone is dropped into quiet lake and waves more in a circle at a speed of 3.5 cm/sec. At the instant when the radius of the circle wave is 7.5 cm, how fast is the enclosed area increasing?

16. A particle moves along the curve $6y = x^3 + 2$. Find the point

on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.

17. The volume of a cube is increasing at a constant rate. Prove that the increase in surface area varies inversely as the length of the edge of the cube.
18. The length x of a rectangle is decreasing at the rate of 2 cm/sec and the width y is increasing at the rate of 2 cm/sec. When $x = 12$ cm and $y = 5$ cm, find the rate of change of (i) the perimeter (ii) the area of the rectangle.
19. A man 2 meters high, walks at a uniform speed of 6 meters per minute away from a lamp post, 5 meters high. Find the rate at which the length of his shadow increases.
20. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground away from the wall at the rate of 2 m/sec. How fast its height on the wall decreasing when the foot of the ladder is 4m away from the wall?
21. The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm/sec. How fast is the area decreasing when the two equal sides are equal to the base?
22. Sand is pouring from a pipe at the rate of 12 cm^3/sec. The falling sand forms a cone on the ground in such away that the height of the cone is always one-sixth of the falling of the base. How fast is the height of the sand-cone increasing when the height is 4 cm?
23. An inverted cone has a depth of 10 cm and a base of radius 5 cm. Water is poured into it at the rate of 3/2 c.c. per minute. Find the rate at which the level of water in the cone is rising when the depth is 4 cm.
24. Water is dripping out from a conical funnel at a uniform rate of 4cm^3/sec through a tiny hole at the vertex in the bottom. When the slant height of the water is 3 cm, find the rate of decrease of

the slant height of the water-cone. Given that the vertical angle of the funnel is 120^0.

25. A water tank has slope of an inverted right circular cone with its axis vertical and vertex lower most. Its semi-vertical angle is $\tan^{-1}(0.5)$. Water is poured into it at a constant rate of 5 cubic metre per hour. Find the rate at which the level of the water is rising at the instant when the depth of water in the tank is 4m.
26. The radius of a circle is increasing at the rate of 0.7 cm/sec. What is the rate of increase of its circumference?
27. A balloon which always remains spherical, is being inflated by pumping in 900 cubic centimeters of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15cm.
28. A stone is dropped into a quiet lake and waves moves move in circles at a speed of 4 cm/sec. At the instant when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?
29. The surface area of a spherical bubble is increasing at the rate of 2 cm^2/sec and its altitude is decreasing at the rate of 3 cm/sec. Find the rate of change of volume when radius is 3 cm and altitude 5 cm.
30. Find the point on the curve $y^2 = 8x$ for which the abscissa and ordinate change at the same rate.
31. The volume of a cube is increasing at the rate 9cm^3/sec. How fast is the surface area increasing when the length of an edge is 10 cm?
32. The volume of a spherical balloon is increasing at the rate of 25cm^3/sec. Find the rate of change of its surface area at the instant when radius is 5 cm.

33. The length x of a rectangle is decreasing at the rate of 5cm/ minute and the width y is increasing at the rate of 4 cm/minute. When $x = 8$ cm and $y = 6$, find the rates of change of (i) the perimeter (ii) the area of the rectangle.
34. A circular disc of radius 3 cm is being heated. Due to expansion, its radius increasing at the rate of 0.05 cm/sec. Find the rate at which its area is increasing when radius is 3.2 cm.
35. The amount of pollution content added in air in a city due to x diesel vehicles is given by $p(x) = 0.005x^3 + 0.02x^2 + 30x$. Find the marginal increase in pollution content when 3 diesel vehicles are added and write which value is indicated in the above questions.

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11 Differentials, Errors and Approximations

Definition

let $y = f(x)$ be any function of x . let Δx be the small change in x and Δy be the corresponding change in y , i.e, $\Delta y = f(x + \Delta x) - f'(x)dx$ or $dy = \frac{dy}{dx}$. Δx is a good approximation of Δy , when $dx = \Delta x$ is relatively small , we denote it by $dy \approx \Delta y$

Thus, $f((x + \Delta x) \approx f(x) + f'(x).C$

Some IMP terms

(i) Absolute error

The error . Δx in x, is called the absolute error in x.

(ii) Relative error

If Δx is an error in x , then $\frac{\Delta x}{x}$ is called the relative error in x .

(iii) Percentage error If Δx is an error in x , then $\frac{\Delta x}{x} \times 100$ is called the percentage error in x .

Practice problems

1. If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.
2. Find the approximate value of $f(3.02)$ .where $f(x) = 3x^2 + 5x + 3$.

On finding the approximate value using differential algorithm

Step1. Define a functional relationship between the independent variable x and dependent variable y by observing the given expression.

Step2. Choose the value of x nearest to the value for which we have to find y in such a way that either y is given for the chosen x or y can be easily computed for chosen x.

Step3. Denote the value of x at which we have to find $y \ by \ x+\Delta x$.

Step4. Find Δx . and assume that $dx = \Delta x$.

Step5. Find dy/dx from the relation in step 1.

Step6. Find $dy = \frac{dy}{dx}.dx$

Step7. Assume that $\Delta y \cong dy$.

Step8. Find the value of y by putting the value of x choose in step ii in the relation obtained by step i.

Step9 The approximate value of y is $y+\Delta y$.

Practice Problem

3. Using differentials, find the approximate values of the following :

I. $\sqrt{26}$	II. $(66)^{1/3}$	III. $\sqrt{0.48}$
IV. $(82)^{1/4}$	V. $\sqrt{49.5}$	VI. $(3.968)^{3/2}$

4. Find the approximate value of $f(2.01)$, where $f(x) = 4x^2 + 5x + 2$.

5. Find the approximate value of $f(5.001)$, where $f(x) = x^3 - 7x^2 + 15$

6. Find the approximate value of $\log_{10} 1005$, given that $\log_{10} e = 0.4343$.
7. If the radius of a sphere is measured as 9 cm with an area of 0.03 m, find the approximate error in calculating its surface area.
8. Find the approximate change in the surface area of a cube of side x metres caused by decreasing the side by 1%.
9. If the radius of a sphere is measured as 7 m with an error of 0.02 m, find the approximate error in calculating its volume.
10. Find the approximate change in the volume of a cube of side x metres caused by increasing the side by 1%.

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12 Mean Value Theorems

Rolle's theorem

Statement

Let f be a real valued function defined on the closed interval [a,b] such that

(i) It is continuous on the closed interval [a,b],

(ii) It is differentiable on the open interval (a,b) ,

(iii) $f(a) = f(b)$

Then there exist a real number $c \in (a,b)$ such that $f'(c) = 0$.

On verification of rolles theorem for a given function on a given function defined on a given interval

Practice problems

1. Verify Rolle's theorem for each of the following function on the indicated interval

(i) $f(x) = x^2 - 5x + 6$ on [2, 3].

(ii) $f(x) = x^3 - 6x^2 + 11x - 6$ on [1,3]

(iii) $f(x) = \sqrt{4 - x^2}$ on [-2,2]

(iv) $f(x) = \sin x + \cos x - 1$ on $[0, \pi / 2]$

(v) $f(x) = e^x(\sin x - \cos x)$ on $[\pi / 4, 5\pi / 4]$

(vi) $f(x) = x^2 - 4x + 3$ on [1, 3]

(vii) $f(x) = x^2 - 5x + 4$ on [1, 4]

(viii) $f(x)=\log(x^2+2)-\log 3$ on [-1,1]

2. Using Rolle's theorem, find points on the curve $y=16-x^2, x\in[-1,1]$, where tangent is parallel to x-axis.
3. If $f:[-5,5]\to R$ is differentiability and if $f'(x)$ doesn't vanish anywhere, than prove that $f(-5)\neq f(5)$.
4. Examine if Rolle's theorem is application to any one of the following functions :

   I. $f(x)=[x]$ *for* $x\in[5,9]$

   II. $f(x)=[x]$ *for* $x\in[-2,2]$
5. It is given that for the function f $f(x)=x^3-6x^2+ax+b$ on [1,3] Rolle's theorem holds with $c=2+\frac{1}{\sqrt{3}}$. find the values of a & b , if $f(1)=f(3)=0$
6. It is given that for the function $f(x)=x^3+bx^2+ax$ on $x\in[1,3]$ Rolle's theorem holds with $c=2+\frac{1}{\sqrt{3}}$. find the values of a & b.
7. Find the point on the curve $y=\cos x-1$, $x\in,\left[\frac{\pi}{2},\frac{\pi}{3}\right]$ at which the tangent is parallel to the x axis.

Langranges mean value theorem

Statement

Let $f(x)$ be a function defined on [a,b] such that

(i) It is continuous on [a,b]

(ii) It is differentiable on (a,b).

Then there exist a real number $c \in (a,b)$ such

that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Verification of langranges mean value theorem

Practice problems

1. Using Langrange's mean value theorem , find a point on the curve $y = \sqrt{x-2}$ defined on the interval [2,3], where the tangent is parallel to the chord joining the end points of the curve.

   Ans. $\left[9/4, \tfrac{1}{2}\right]$

2. Using Lagrange's mean value theorem, show that $\sin x < x$ for $x < 0$.
3. Using Lagrange' s mean value theorem, prove that $\dfrac{b-a}{b} < \log\left(\dfrac{b}{a}\right) < \dfrac{b-a}{a}$, where $0 < a < b$.
4. Let f and g be differentiable on [0, 1] such that f (0) = 2, g (0) = 0, f (1) = 6 and g (1) = 2. Show that there exists $c \in (0,1)$ such that f ' (c).
5. Verify Lagrange 's mean value theorem for the following functions on the indicated intervals. In each case find a point 'c' in the indicated interval as stated by the Lagrange 's mean value theorem :

   (i) $f(x) = x + \dfrac{1}{x}$ on [1, 3]

   (ii) $f(x) = x(x+4)^2$ on [0, 4]

(iii) $f(x) = \sqrt{x^2 - 4}$ on [2, 4]

(iv) $f(x) = x^3 - 5x^2 - 3x$ on [1, 3]

(v) $f(x) = \sin x - \sin 2x - x$ on [0,ð]

(vi) $f(x) = x^2 - 2x + 4$ on [1,5]

6. Let C be a curve defined parametrically as $x = a\cos^3\theta$ $0 \le \theta \le \frac{\pi}{2}$. Determine a point P on C, where the tangent to C is parallel to the chord joining the points (a, 0) and (0, a).

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13 | Tangents And Normals

Tangent and normal

Tangent

A line which touches the curve at a single point is called tangent at a point .

Normal

if a line is perpendicular to the tangent at a point of contact , then it is called normal.

slope of tangent and normal

(i) The slope of a tangent at the point of contact to the curve $y = f(x)$ at the point (x_1, y_1) is given by $\left(\frac{dy}{dx}\right)_{(x1,y1)}$ or $f'(x_1)$

(ii) The slope of a normal to the curve $y = (x)$ at the point (x_1, y_1) is given by $-1/\left(\frac{dy}{dx}\right)_{(x1,y1)}$ or $f'(x_1)$ .or $-\left(\frac{dy}{dx}\right)$.

Practice problems

1. Show that the tangents to the curve $y = 2x^3 - 3$ at the points where $x = 2$ and $x = -2$ are parallel.
2. Prove that the tangents to the curve $y = x^2 - 5x + 5$ at the points (2, 0) and (3, 0) are at right angles.
3. Find the slope of the normal to the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$.

4. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.
5. Find the points on the curve $y = x^3 - 11x + 5$ at which the tangent has the equation $y = x - 11$.
6. If the tangent to the curve $y = x^3 + ax + b$ at (1, -6) is parallel to the line $x - y + 5 = 0$, find a and b.
7. At what point of the curve $y = x^2$ does the tangent make an angle of 45^0 with the x-axis?
8. Find the points on the curve $\frac{x^2}{4} + \frac{y^2}{25} = 1$ at which the tangents are parallel to the (i) x-axis. (ii) y-axis.
9. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the (i) x-axis. (ii) y-axis.
10. Find the points on the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at which the tangents are (i) parallel to a-axis (ii) parallel to y-axis.
11. Show that the tangents to the curve $y = 7x^3 + 11$ at the point $x = 2$ and $x = -2$ are parallel.
12. Find the points on the curve $y = x^3$ where the slope of the tangent is equal to x-coordinate of the point.

Equation of tangent and normal

(1) The equation of tangent to the curve $y = f(x)$ at the point $P(x_1, y_1)$ is given by $y - y_1 = m(x - x_1)$ where $m = dy / dx$ at a point (x_1, y_1)

(2) The slope of a normal to the curve $y = f(x)$ at the point $Q(x_1, y_1)$ is given by $y - y_1 = \frac{-1}{m}(x - x_1)$ where m=dy/dx at point at (x_1, y_1).

Algorithm

Step1. Find $\frac{dy}{dx}$ from the given equation $y = f(x)$.

Step 2. Find $\frac{dy}{dx}$ at the point $P(x_1, y_1)$.

Step3. If $\left[\frac{dy}{dx}\right](x_1, y_1)$. Is a non zero finite number, then obtain the equation of tangent and normal at (x_1, y_1)

Step 4. If $\left[\frac{dy}{dx}\right](x_1, y_1) = 0$, then the equation of the tangent and normal at (x_1, y_1) are $y - y_1 = 0$ and $x - x_1 = 0$ respectively. If $\left[\frac{dy}{dx}\right](x_1, y_1) = \pm\infty$, then the equation of the tangent and normal at (x_1, y_1) are $y - y_1 = 0$ and $x - x_1 = 0$ respectively.

Some Important Terms

1. If slope of the tangent is zero, then $\tan\theta = 0$, so $\theta = 0$, which means that line is parallel to the X- axis and then equation of tangent at the point (x_1, y_1) is $y = y_1$

2. If $\theta \to \frac{\pi}{2}$ then $\tan\theta \to \infty$ which means that tangent line is perpendicular to the X-axis, i.e, parallel to the Y axis and then equation of the tangent at the point (x_1, y_1) is $x = x_1$

3. Suppose m_1 and m_2 are slopes of tangent to the curves. The condition for two curves to be perpendicular (orthogonal), is $m_1m_2 = -1$.
4. The angle of the intersection between two curves is the angle between the tangents to the curves at the point of intersection.

Practice problems

1. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.
2. Find the equation of the tangent line to the curve $x = 1 - \cos\theta$, $y = \theta - \sin\theta$ at $\theta = \pi/4$.
3. Find the equations of the tangent and the normal to the point 't' on the curve $x = a \sin^3 t$, $y = \cos^3 t$.
4. Show that the line $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point where it crosses the y-axis.
5. Find the equations of the tangent and the normal to the curve $y = \frac{x-7}{(x-2)(x-3)}$ at the point, where it cuts x-axis.
6. Find the equation (s) of tangent(s) to the curve $y = x^3 + 2x + 6$ which is perpendicular to the line $x + 14y + 4 = 0$.
7. Find the equations of the normal to the curve $x^2 = 4y$ which passes through the point (1, 2).
8. For the curve $y = 4x^3 - 2x^5$ find all points all which the tangent passes through the origin.
9. Prove that all normals to the curve $x = a\cos t + at \sin t$ at

y=a sin t y=a sin t -at cos t are at a distance a from the origin.

10. Find the equations of the tangent and the normal to the following curves at the indicated points:

    I. $y = x^4 - bx^3 + 13x^2 - 10x + 5$ at (0,5)

    II. y = x-6x+13x-10x+5 atx=1.

    III. $y = x^2 + 4x + 1$ at x=3

    IV. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\ at(x_{0,}y_0)$

    V. $x^{2/3} + y^{2/3} = 2$ at(1,1)

    VI. $4x^2 + 9y^2 = 36\ at(3\cos\theta, 2\sin\theta)$

    VII. $y^2 = 4ax$ at (x_1, y_1)

    VIII. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 at(\sqrt{2a}, b)$

11. Find the equation of the normal to the curve $ay^2 = x^3$ at the point (am^2, am^3).

12. Find the equation of the normal to the curve $y = x^3 + 2x + 6$ which is parallel to the line $3x - y + 1 = 0$.

13. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is

    I. Parallel to the line $2x - y + 9 = 0$

    II. Perpendicular to the line $5y - 15x = 13$.

14. Find the equations of all lines having slope 2 and that are tangent to the curve $y = \frac{1}{x-3}, x \neq 3$.

15. Find the equations of all lines of slope zero and that are tangent to the curve $y = \dfrac{1}{x^3 - 2x + 3}$.

16. Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$.

17. Find the equation of the normal to the curve $x^2 + 3y - 3 = 0$, which is parallel to the line $y = 4x - 5$.

18. Prove that $\left(\dfrac{x}{a}\right)^n + \left(\dfrac{y}{b}\right)^n = 2$ touches the straight line $\dfrac{x}{a} + \dfrac{y}{b} = 2$ for all $n \in N$ at the point (a,b).

19. Find the equation of the normal to the curve $x = \sin 3t$, $y = \cos 2t$ at $t = \dfrac{\pi}{4}$.

20. At what points will be tangent to the curve $y = 2x^3 - 15x^2 + 36x - 21$ be parallel to x-axis ?

21. Find the equation of the normal to the curve $3x^2 - y^2 = 8$, which passes through the point $(4/3, 0)$.

22. Show that the curves $x = y^2$ and $xy = k$ cut at right angles, if $8k^2 = 1$.

23. Find the values of p for which the curves $x^2 = 9p(9 - y)$ and $x^2 = p(y+1)$ cut each other at right angles.

24. Show that the curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other.

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14 Increasing And Decreasing Functions

Defination

1. Let I be an open interval contained in the domain of a real valuedfunction f. then, f is said to be

(i) Increasing on I if $x_1 < x_2 in\ I \Rightarrow f(x_1) \leq f(x_2), \forall x_1, x_2 \in I$

(ii) Strictly increasing on I,if

$x_1 < x_2 in\ I \Rightarrow f(x_1) < f(x_2), \forall x_1, x_2 \in I$

(iii) Decreasing on I, if $x_1 < x_2 in\ I \Rightarrow f(x_1) \geq f(x_2), \forall x_1, x_2 \in I$

(iv) Strictly decreasing on I if

$x_1 < x_2 in\ I \Rightarrow f(x_1) > f(x_2), \forall x_1, x_2 \in I$

2. Let f be a continuous on [a,b] and differentiable on the open interval (a,b) . then,

(i) f is increasing in [a,b] , if $f'(x) > 0$ for each $x \in (a,b)$.

(ii) f is decreasing in [a,b] , if $f'(x) < 0$ for each $x \in (a,b)$.

(i) f is constant function in [a,b] , if $f'(x) = 0$ for each $x \in (a,b)$

Monotonic Function

A function which is either increasing or decreasing in a function in a given interval I, is called monotonic function.

On finding the intervals in which a function is increasing or decreasing

Algorithm

Step1**.** Obtain the function and put it equal to f(x).

Step 2. Find f'(x)

Step3. Put $f'(x) > 0$ and solve this inequation.

For the values of x obtained in step III $f(x) = 0$ is increasing and for the remaining points in its domain it is decreasing.

Practice problems

1. Find the intervals in which the function $f(x) = 2x^3 - 9x^2 + 12x + 15$ is (i) increasing, (ii) decreasing:
2. Find the intervals in which $f(x) = (x+1)^3(x-3)^3$ is increasing or decreasing.
3. Find the intervals in which the function $f(x) = \log(1+x) - \dfrac{2x}{2+x}$ is increasing or decreasing.
4. Find the intervals in which $f(x) = \dfrac{4x^2+1}{x}$ is increasing or decreasing.
5. Find the intervals in which the function f given by $f(x) = x^3 + \dfrac{1}{x^3}, x \neq 0$ is I. Increasing II. Decreasing
6. Separate $[0, \pi, /2]$ into subintervals in which $f(x) = \sin 3x$ is increasing or decreasing.
7. Find the intervals in which the function f given by $f(x) = \sin x + \cos x, 0 \leq \times \leq 2\pi$ is increasing or decreasing.
8. Find the intervals in which the function f given by $f(x) = \dfrac{4\sin x - 2x - x\cos x}{2 + \cos x}$, $0 \leq \times \leq 2\pi$ (i) increasing (ii) decreasing
9. Separate the interval $[0, \pi, /2]$ into sub-intervals in which $f(x) = \sin^4 x + \cos^4 x$ is increasing or decreasing.

10. Prove that the function $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing on R.

11. Prove that $f(\theta) = \dfrac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function of θ in $\left[0, \dfrac{\pi}{2}\right]$.

12.Which of the following functions are decreasing on $[0, \pi, /2]$

I. $\cos x$

II. $\cos 2x$

III. $\tan x$

IV. $\cos 3x$

13. Prove that the function $f(x) = x^2 - x + 1$ is neither increasing nor decreasing on $(-1,1)$.

14. On which of the following intervals is the function $f(x)x^{100} + \sin x - 1$ increasing ?

I. $(0, \pi / 2)$

II. $(\pi / 2, \pi)$

III. (0, 1)

IV. (-1, 1)

15. Find the least value of 'a' such that the function $f(x) = x^2 + ax + 1$ is increasing on [1, 2].

16. Find the intervals in which the following functions are increasing or decreasing.

I. $f(x) = 10 - 6x - 2x^2$

II. $f(x) = x^2 + 2x - 5.$

III. $f(x) = 6 - 9x - x^2$

IV. $f(x) = 2x^3 + 9x^2 + 12x + 20$

V. $f(x) = -2x^3 - 9x^2 - 12x + 1$

VI. $f(x) = x^3 - 12x^2 + 36x + 17$

VII. $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x + \frac{36}{5}x + 11$

VIII. $f(x) = x^3 - 6x^2 + 9x + 15$

IX. $f(x) = \{x(x-2)\}^2$

X. $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

XI. $f(x) = \frac{3}{2}x^4 - 4x^3 - 45x^2 + 51$

XII. $f(x) = \log(2+x) - \frac{2x}{2+x}, x \in R$.

17. Show that $f(x) = e^{2x}$ is increasing on R.

18. Prove that the function $f(x) = x^3 - 6x^2 + 12x - 18$ is increasing on R.

19. Find the intervals in which $f(x) = \log(1+x) - \frac{x}{1+x}$ is increasing or decreasing.

20. Prove that the function f given by $f(x) = \log \cos x$ is strictly increasing on $(-\pi/2, 0)$ and strictly decreasing on $(0, \pi/2)$.

21. Prove that the function $f(x) = \cos x$ is :

I. Strictly decreasing in $(0, \pi)$

II. Strictly increasing in $(\pi, 2\pi)$

III. Neither increasing nor decreasing in $(0, 2\pi)$

22. Let f defined on [0, 1] be twice differentiable such that $|f''(x)| \leq 1$ for all x $\in [0,1]$. If f(0)=f(1), then show that $|f'(x)| \leq 1$ for all x $\in [0,1]$.

23. Find the intervals in which f (x) is increasing or decreasing:

I. $f(x) = x|x|, x \in R$

II. $f(x) = \sin x + |\sin x|, 0 < x < 2\pi$

III. $f(x) = \sin x(1 + \cos x), 0 < x < \dfrac{\pi}{2}$

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15 | Maxima and Minima

Maximum and minimum value

Let f be a function define on an interval I . then ,

1. f is said to have a maximum value in I, then, there exist a point c in I such that $f(c) \geq f(x)$, $\forall x \in i$. The number $f(c)$ is called the max. value of f in I and the point c is called **a point of maximum value** of f in I.
2. f is said to have a minimum value in I, then , there exist a point c in I such that $f(c) \leq f(x)$, $\forall x \in i$. The number $f(c)$ is called the max. value of f in I and the point c is called a **point of minimum value** of f in I.
3. f is said to have an extreme value in I, then, there exist a point c in I such that $f(c)$ is either a maximum or a minimum value of f in I The number $f(c)$ is called an extreme value of f in I and the point c is called an extreme point.

Local maxima and local minimum

local maximum

A function $f(x)$ is said to attain a local maximum at $x = a$, if there exist a negihbourhood $(a-\delta, a+\delta)$ of a such that

$f(x) < f(a)$ *for all* $x \in (a-\delta, a+\delta), x \neq a$.

$f(x) - f(a) < 0$ *for all* $x \in (a-\delta, a+\delta), x \neq a$.

In such a case , $f(a)$ is called the local maximum value of $f(x)$ at $x = a$.

Local minimum

A function $f(x)$ is said to attain a local minimum at $x = a$, if there exist a negihbourhood $(a-\delta, a+\delta)$ of a such that

$f(x) > f(a)$ *for all* $x \in (a-\delta, a+\delta), x \neq a$.

$f(x) - f(a) > 0$ *for all* $x \in (a-\delta, a+\delta), x \neq a$.

In such a case , f(a) is called the local minimum value of $f(x)$ at $x = a$.

4. Find the maximum and the minimum values, if any of the following functions
   (i) $f(x) = \sin 3x + 4, x \in (-\pi/2, \pi/2)$
   (ii) $f(x) = x^3 + 1$ *for all* $x \in R$.
5. Find the maximum and the minimum values, if any, without using derivatives of the following functions:
   I. $f(x) = |\sin 4x + 3|$ *on* R
   II. $f(x) = -|x+1| + 3$ *on* R
   III. $f(x) = -(x-1)^2 + 2$ *on* R
   IV. $f(x) = \sin 2x + 5$ *on* R

First derivative test of local maxima and minima

Algorithm

Step1. Put $y = f(x)$

Step2. Find dy/dx

Step3. Put $dy/dx = 0$ and colve this equation for x. Let c_1, c_2, c_3….c_n be the roots of this equation . Points c_1,c_2, c_3…..c_n are critical points Now we test the function at each one of these points.

Step4. Consider $x = c_1$.

If dy / dx changes its sign from positive to negative as x increase through c_1, then the function attains a local maximum at $x = c_1$.

If dy / dx changes its sign from negative to positive as x increase through c_1, then the function attains a local minimum at $x=c_1$.

If dy / dx does not chang its sign from as x increase through c_1, then $x = c_1$ is neither a point of local minimum nor maximum. In this case $x = c_1$ is a point of inflexion.

Similarly , we may solve eith other values of x.

Practice Problems

6. Find the points of local maxima or local minima, if any of the following functions, using the first derivative test. Also, find thelocal maximum or local minimum values, as the case may be-

    I. $f(x) = \dfrac{1}{x^2 + 2}$

    II. $f(x) = x^3 - 3x$

    III. $f(x) = \sin x - \cos x, 0 < x < 2\pi$

    IV. $f(x) = x\sqrt{1-x}, x > 0$

    V. $f(x) = x^3 - 6x^2 + 9x - 8$

7. Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ has neither maxima nor minima

Higher Order Derivative Test

Algorithm

Step1. Put $y = f(x)$

Step2. Find dy/dx

Step3. Put $dy/dx = 0$ and solve this equation for x. Let $c_1, c_2, c_3 \ldots$.cn be the roots of this equation. Points $c_1, c_2, c_3 \ldots$..c_n are critical points Now we test the function at each one of these points.

Step4. Find $f''(x)$. Consider $x = c_1$.

If $f(c_1) < 0$, then $x = c_1$ is a point of local maximum.

If $f(c_1) > 0$, then $x = c_1$ is a point of local minimum.

If $f(c_1) = 0$, we must find $f''(x)$ and substitute in it $c1$ for x.

If $f''(c_1) = 0$, we must find $f''(x)$ and substitute in it $c1\, for\, x$.

If $f''(c_1) \neq 0$, then $x = c1$ is neither a point of local maxima nor minima and it is called point of inflection.

8. Find the points of local maxima or local minima, if any, of the following functions. Find also the local maximum or local minimum values, as the case may be:

   I. $f(x) = \sin x + \cos x$, where $0 < x \frac{\pi}{2}$

   II. $f(x) = \sin x - \cos x$, where $0 < x < 2\pi$

9. Show that none of the following functions has a local maximum or a local minimum:

I. $x^3 + x^2 + x + 1$

II. e^x

III. $\log x$

IV. $\cos x, 0 < x < \pi$

10. At what points, the slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ at is maximum ? Also, find the maximum slope.

11. It is given that at x = 1, the function $x^4 - 62x^2 + ax + 9$ attains its maximum values on the interval [0, 2]. Find the value of a.

12. Show that $\frac{\log x}{x}$ has a maximum value at $x = e$.

Maximum And Minimum Values in a Closed Interval

Algorithm

Step1. Find $f'(x)$

Step2. Put $f'(x) = 0$ and find values of x. let $c_1, c_2, c_3 \ldots c_n$.

Step3. Take the maximum and minimum values out of the values

$f(a) f(c_1) f(c_2) \ldots . f(c_n) . f(b)$

Practice problems

13. Find the maximum and minimum values of $f(x) = 2x^3 - 24x + 107$ in the interval [1, 3]

14. Find the maximum and minimum values of $f(x) = x + \sin 2x$ in the interval $[0, 2\pi]$.

15. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:

(i) $f(x) = (1/2 - x)^2 + x^3$ in [-2, 2.5].

(ii) $f(x) = \sin x + \cos x$ in $[0, \pi]$

(iii) $f(x) = 4x - \frac{x^2}{2}$ in [-2,4.5]

(iv) $f(x) = (x-2)\sqrt{x-1}$ in [1,9]

(v) $3x^4 - 8x^3 + 12x^2 - 48x + 25$ in [0,3]

Applied problem on maxima and minima

In this section , following result very useful.

(i) For a square of side x Area = x^2, parameter = $4x$.

(ii) For a rectangle of sides x& y: Area xy, Perimeter = $2(x+y)$

(iii) For a circle of radius r Area = πr^2, Circumference = $2\pi r$

(iv) For a sphere of radius r Volume = $4/3\pi r^3$,Surface area = $4\pi r^2$

(v) For a right circular cone of height h , slant height l and radius of the base r: Volume = $1/3\pi r^3 h$, Curved surface= πrl , total surface = $\pi r^2 + \pi rl$

(vi) For a right circular cylinder of height h , and radius of the base r: Volume = $\pi r^2 h$, Surface = $2\pi rh + 2\pi r^2$

Curved surface = $2\pi rh$

(vii) For a cube of length x Volume = x^3 ,Surface area = $6x^2$

(viii) For a cuboid of edge length x, y and z

Volume = xyz, Surface area = $2(xy = yz + zx)$

(ix) Area of equilateral triangle $= \frac{\sqrt{3}}{4}(side)^2$

Practice problems

1. Find two numbers whose sum is 24 and whose product is as large as possible.
2. Find two positive numbers x and y such that their sum is 35 and the product $x^2 y^5$ is maximum.
3. Find the minimum value of ax + by, where $xy = c^2$ and a, bb, c are positive.
4. Show that of all the rectangles of given area, the square has the smallest perimeter.
5. Show that of all rectangles inscribed in a given circle, the square has the maximum area.
6. Show that the rectangle of maximum perimeter which can be inscribed in a circle of radius a is a square of side $\sqrt{2}a$.
7. Tangent to the circle $x^2 + y^2 = a$ at any point on it in the first quadrant makes intercepts OA and OB on x and y axes respectively, O being the centre of the circle. Find the minimum value of OA + OB.
8. If the sum of the lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angel between them is $\pi / 3$.
9. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of the material will be least when depth of the tank is half of its width.
10. An open box with square base is to be made out of a given quantity of card board of area c^2 square units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units.

11. Show that a cylinder of a given volume which is open at the top, has minimum total surface area, provided its height is equal to the radius of its base.
12. Show that the height closed cylinder of given surface and minimum volume, is equal to the diameter of its base.
13. Show that the height cylinder, which is open at the top, having a given surface area and greatest volume, is equal to the radius of its base.
14. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a is $\frac{2a}{\sqrt{3}}$.
15. Show that the semi-vertical angle of a cone of maximum volume and given slant height is $\tan^{-1}\sqrt{2}\ \ or \cos^{-1}\frac{1}{\sqrt{3}}$.
16. Show that the volume of the largest cone that can be inscribed in a sphere of radius R is 8/27 of the volume of the volume of the sphere.
17. Prove that the radius of the right circular of greatest curved surface which can be inscribed in a given cone is half of that of the cone.
18. Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle α is $\frac{4}{27}\pi h^3 \tan^2\alpha$. Also, show that height of the cylinder is $\frac{h}{3}$.
19. Let AP and BQ be two vertical poles at points A and B respectively. If AP = 16 m, BQ = 22 m and AB = 20 M, then find the distance of a point R on AB from the point A such that $rp^2 + rq^2$ is minimum.

20. If the length of three sides of a trapezium other than base are equal to 10 cm, then find the area of trapezium when it is maximum.
21. Find the shortest distance of the point (0, c) form the parabola $y = x^2$ where $0 \le c \le 5$.
22. Find the area of the greatest isosceles triangle that can be inscribed in a given ellipse having its vertex coincident with one end of the major axis.
23. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
24. A point on the hypotenuse of a right triangle is at distances a and b from the sides of the triangle. Show that the minimum length of the hypotenuse is $(a^{2/3} + b^{2/3})^{3/2}$.
25. A wire of length 28 m is to be cut into pieces. One of the pieces is to be made into a square and the other into a circle. What should be the lengths of the two pieces so that the combined area of the circle and the square is minimum?
26. Given the sum of the perimeters of a square and a circle, show that the sum of their areas is least when one side of the square is equal to diameter of the circle.
27. Find the largest possible area of a right angled triangle whose hypotenuse is 5 cm long.
28. Two sides of a triangle have lengths 'a' and 'b' and the angle between them is è. When value of è will maximize the area of the triangle? Find the maximum area of the triangle also.
29. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs Rs. 70 per square metre for the

base and Rs. 45 per square metre for sides, what is the cost of least expensive tank?

30. A window in the form of a rectangle is surmounted by a semi-circular opening. The total perimeter of the window is 10 m. Find the dimensions of the rectangular part of the window to admit maximum light through the whole opening.
31. Prove that a conical tent of given capacity will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base.
32. Show that the height of the cone of maximum volume that can be inscribed in a sphere of radius 12 cm is 16 cm.
33. A close cylinder has volume 2156 cm^3. What will be the radius of its base so that its total surface area is minimum ?
34. Show that the maximum volume of the cylinder which can be inscribed in a sphere of radius $5\sqrt{3}$ cm is $500\pi\, cm^3$.
35. Find the point on the curve $x^2 = 8y$ which is nearest to the point (2, -8).
36. Find the coordinates of a point on the parable $y = x^2 + 7x + 2$ which is closest to the straight line $y = 3x - 3$.
37. Find the maximum slope of the curve $y = -x^3 + 3x^2 + 2x - 27$.
38. Manufacture can sell x items at a price of $\left(5 - \frac{x}{100}\right)$Rs. . The cost price is Rs. $\left(\frac{x}{5} + 500\right)$. Find the number of items he should sell to earn maximum profit.

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16 Indefinite Integrals

Definition

Integration is the inverse process of differentiation. Thus

if $\frac{d}{dx}[F(x)] = f(x)$, then $\int f(x)$ dx =F(x)+C

Properties of Indefinite Integral

1. $\int [f(x) \pm g(x)dx] = \int f(x)dx \pm \int g(x)dx$
2. for any real number k $\int kf(x)dx = k\int f(x)dx$

Basic formulae of integrals

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$
2. $\int e^{ax} dx = \frac{e^{ax}}{a} + c$
3. $\int a^x dx = \frac{a^x}{\log a} + c$
4. $\int \frac{1}{x} dx = \log x + C$
5. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \quad n \neq -1$
6. $\int \sin x dx = -\cos x + c$
7. $\int \cos x dx = \sin x + c$
8. $\int \tan x dx = -\log(\cos x) + c$

9. $\int \cot x dx = \log(\sin x) + c$

10. $\int \sec x dx = \log(\sec x + \tan x) + c$

11. $\int \cos ecx dx = \log(\cos ecx - \cot x) + c = \log(\tan x / 2) + c$

12. $\int \sec x \tan x dx = \sec x + c$

13. $\int \cos ecx dx$ cotxdx=-cosecx+c

14. $\int \sec^2 x dx = \tan x + c$

15. $\int \cos ec^2 x dx = -\cot x + c$

16. $\int \frac{1}{\sqrt{1-x}^2} dx = \sin^{-1} x + c$

17. $\int \frac{-1}{\sqrt{1-x}^2} dx = \cos^{-1} x + c$

18. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

19. $\int \frac{-1}{1-x^2} dx = \cot^{-1} x + c$

20. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$

21. $\int \frac{-1}{x\sqrt{x^2-1}} dx = co\sec^{-1} x + c$

22. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}(x/a) + c$

23. $\int \sqrt{x^2 - a^2}dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right) + c$

24. $\int \sqrt{x^2 + a^2}dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left(x + \sqrt{x^2 + a^2}\right) + c$

25. $\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(x / a\right) + c$

Practice problems

Evaluate the following integrals:

1. $\int x^4 dx$
2. $\int \sqrt{x}dx$
3. $\int \frac{1}{\sqrt{x}}dx$
4. $\int \sqrt[5]{x}dx$
5. Evalate : $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}dx$
6. $\int 4x^5 dx$
7. $\int 3^{x+2} dx$
8. $\int \frac{1}{2}\sec^2 x dx$
9. $\int \sqrt{1 - \sin 2x}dx$
10. $\int \frac{\cos x - \cos 2x}{1 - \cos x}dx$
11. $\int \cot^2 x dx$

12. $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha} dx$

13. $\int \frac{\sin^6 x - \cos^6 x}{s\,in^2 x \cos^2 x} dx$

14. $\int \tan^{-1}\left\{\sqrt{\left(\frac{1-\cos 2x}{1+\cos 2x}\right)}\right\} dx, \quad 0 < x < \pi/2$

15. $\int \tan^{-1}(\sec x + \tan x) dx, \quad -\pi/2 < x < \pi/2$

16. $\int \tan^{-1}\left\{\sqrt{\frac{1-\sin x}{1+\sin x}}\right\} dx, \quad -\pi/2 < x < \pi/2$

17. $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx.$

18. $\int (\tan x + \cot x)^2 dx$

19. $\int \frac{1-\cos x}{1+\cos x} dx.$

20. $\int \frac{\cos x}{1+\cos x} dx.$

21. $\int e^{2x-3} dx$

22. $\int \sec^2(7-4x) dx$

23. $\int co\sec^2(3x+2) dx.$

24. $\int \sin(ax+b)\cos(ax+b) dx.$

25. $\int \frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x \cos^2 x} dx.$

26. $\int \frac{1+\cos x}{1-\cos x}dx.$

27. $\int \frac{1}{\sqrt{x+3}-\sqrt{x+2}}dx.$

28. $\int \tan^2(2x-3)dx.$

29. $\int \frac{x+2}{(x+1)^2}dx$

30. $\int \frac{2+x+x^2}{x^2(2+x)}+\frac{2x-1}{(x+1)^2}dx$

31. $\int \frac{x^2+1}{(x+1)^2}dx.$

Evaluation of integrals of the form $\int (ax+b)\sqrt{cx+d}\ dx$ and $\int \frac{ax+b}{\sqrt{cx+d}}dx$

In order to evaluate this type of integrals, we may use the following algorithm:

Algorithm

Step1.Write $(ax+b)$ in terms of $(cx+d)$ as follows:

$$(ax+b)=\lambda(cx+d)+\mu$$

Step 2. Find λ and μ by equating coefficients of like powers of x on both sides.

Step3. Replace $ax+b$ by $\lambda(cx+d)+\mu$ in the given integral to obtain.

Practice Problems

1. Evaluate

(i) $\int \frac{x+1}{\sqrt{2x-1}}dx$

(ii) $\int \frac{x}{\sqrt{x+2}} dx$

(iii) $\int (5x+3)\sqrt{2x-1} dx$

(iv) $\int \frac{8x+13}{\sqrt{4x+7}} dx$

(v) $\int (x+2)\sqrt{3x+5} dx$

evaluation of integral of the form $\int \cos^m x dx$ **,** $\int \sin^m x dx$

To evaluate above integral of the form we use following trigonometric identities:

$\sin^2 x = \frac{1-\cos 2x}{2}$ (ii) $\cos^2 x = \frac{1+\cos 2x}{2}$

(iii) $\sin 3x = 3\sin x - 4\sin^3 x$ (iv) $\cos 3x = 4\cos^3 x - 3\cos x$

2. Evaluate : $\int \cos^2 x dx$

3. Evaluate :

4. Evaluate : $\int \sin^4 x \cos^4 x dx$

5. Evaluate : $\int \cos^4 x dx$

evaluation of integral of the form

$\int \sin mx \cos nx dx$, $\int \sin mx \sin nx dx$,.... $\int \cos mx \cos nx dx$,....

To evaluate above integral of the form we use following trigonometric identities:

(i) $2\sin A \cos B = \sin(A+B) + \sin(A-B)$

(ii) $2\cos A \cos B = \cos(A+B) + \cos(A-B)$

(iii) $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

(iv) $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

6. Evaluate : $\int \cos 2x \cos 4x dx$

7. Evaluate : $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} dx$

8. integrate the following

9. Evaluate : $\int \sin 4x \cos 7x dx$

10. Evaluate : $\int \sin x \cos 2x \sin 3x dx$

integral of the form $\int \frac{f'(x)}{f(x)} dx$

Evaluate:

11. $\int \sqrt{\frac{1+\cos x}{1-\cos x}} dx$

12. $\int \frac{\sin x}{\sin^m (x-a)} dx$

13. $\int \frac{1}{\cos(x-a)\cos(x-b)} dx$

14. $\int \frac{1-\tan x}{1+\tan x} dx$

15. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

16. $\int \frac{\cos 2x}{(\cos x + \sin x)^2} dx$

17. $\int \frac{\sin(x-\alpha)}{\sin(x+\alpha)} dx$

18. $\int \frac{1}{e^2+1} dx$

19. $\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} dx$

20. $\int \frac{1-\sin 2x}{\cos(x+a)\cos(x+b)} dx$

21. $\int \frac{1-\sin 2x}{x+\cos^2 x} dx$

22. $\int \frac{1+\tan x}{x+\log\sec x} dx$

23. $\int \frac{1+\cot x}{x+\log\sin x} dx$

integrals of the form $\int \{f(x)\}^n \{f'(x)\} dx$

put $f(x) = \tan d$ f'(x)dx=dt

Evaluate : $\int \frac{\sec^4 x}{\sqrt{\tan x}} dx$

24. $\int \frac{(1+\log x)^2}{x} dx$

25. $\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$

26. $\int (4x+2)\sqrt{x^2+x+1}dx$

27. $\int \frac{\cos x - \sin x}{1+\sin 2x} dx$

28. $\int \sec x \log(\sec x + \tan x) dx$

29. $\int \cot x \log \sin x dx$

30. $\int \left(\frac{x+1}{x}\right)(x+\log x)^2 dx$

31. $\int \frac{\cos^5 x}{\sin x} dx$

32. $\int \frac{\sin(\tan^{-1} x)}{1+x^2} dx$

33. $\int \tan^3 2x \sec 2x dx$

Integrals of the form $\int (ax+b)^n P(x)dx, \int \frac{P(x)}{(ax+b)^n} dx,$ where p(x) is a polynomial and n is a positive rational number.

Algorithm

Step 1. Substitute $ax+b=t$ or , $x = \frac{t-b}{a}$ and $dx = 1/a\ dt$

Step 2. Simplify the integrand in terms of t and integrate w.r.t. t by

using $\int (t)^n dt = \int \frac{t^{n+1}}{n+1} + c$

Step 3. Replace t by ax + b

34. Evalute : $\int \frac{1}{x^{1/2} + x^{1/3}} dx$

35. $\int \frac{1}{\sqrt[3]{x+1} + \sqrt{x+1}} dx$

36. $\int \tan^2 x \sec^4 x dx$

37. $\int \tan^4 x dx$

38. $\int \sin^3 x \cos^5 x dx$

39. $\int \frac{\sin^4 x}{\cos^8 x} dx$

40. $\int \frac{1}{\sin x \cos^3 x} dx$

Evaluation of integral by making trigonometric substitution

41. $\int \frac{x^2}{\sqrt{1-x^2}} dx$

42. $\int \frac{1}{x\sqrt{x^4-1}} dx$

43. $\int \frac{x^7}{(1-x^2)^5} dx$

44. $\int \frac{1}{(x^2+2x+2)^2} dx$

45. $\int \frac{x^7}{(a^2-x^2)^5} dx$

Some special integrals

(i) $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}(x/a) + c$

(ii) $\int \frac{1}{\sqrt{x^2-a^2}} dx = \log(x+\sqrt{x^2-a^2}) + c$

(iii) $\int \frac{1}{\sqrt{x^2+a^2}} dx = \log(x+\sqrt{x^2+a^2}) + c$

(iv) $\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}(x/a) + c$

(v) $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + c$

(vi) $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left(\frac{x-a}{x-a}\right) + c$

46. Evaluate : $\int \frac{1}{4+9x^2}dx$

47. $\int \frac{1}{16-9x^2}dx$

48. $\int \frac{1}{\sqrt{9-25x^2}}dx$

49. $\int \frac{1}{3x^2+13x-10}dx$

50. $\int \frac{1}{x^2+4x+8}dx$

51. $\int \frac{1}{x^3+6x+13}dx$

52. $\int \frac{1}{x(x^n+1)}dx$

53. $\int \frac{\sin x+\cos x}{9+16\sin 2x}dx$

54. $\int \frac{x}{x^2-x^2+1}dx$

55. $\int \frac{e^x}{(1+e^x)(2+e^x)}dx$

56. $\int \frac{1}{\sqrt{(x-1)(x-2)}}dx$

57. $\int \frac{1}{\sqrt{8+3x-x^2}}dx$

58. $\int \frac{1}{\sqrt{7-3x-2x^2}}dx$

59. $\int \frac{\sec^2 x}{\sqrt{16+\tan^2 x}}dx$

60. $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx$

61. $\int \frac{2x}{\sqrt{1-x^2-x^4}}dx$

62. $\int \frac{\sin x+\cos x}{\sqrt{1+\sin 2x}}dx$

63. $\int \frac{\sec^2 x}{\sqrt{4+\tan^2 x}}dx$

Integrals of the form $\int \frac{P(x)}{ax^2+bx+c}dx$

algorithm

Step1. Write the numerator px+q in the following form:

$$px+q=\lambda\left\{\frac{d}{dx}\left(ax^2+bx+c\right)\right\}+\mu \quad \text{i.e,} \quad px+q=\lambda(2ax+b)+\mu$$

Step2. Obtain the value of ë and ì by equating the coefficient of like power of x on both sides.

Step3. Replace $px+q$ by $\lambda(2ax+b)+\mu$ in the given integral to get

$$\int \frac{px+q}{ax^2+bx+c}dx = \lambda \int \frac{2ax+b}{ax^2+bx+c}dx + \mu \int \frac{1}{ax^2+bx+c} + c$$

Evaluate :

64. $\int \frac{x+2}{2x^2+6x+5}dx$

65. $\int \frac{x^3}{x^4+x^2+1}dx$

66. $\int \frac{(3\sin x-2)\cos x}{5-\cos^2 x-4\sin x}dx$

67. $\int \frac{5x-2}{1+2x3x^2}dx$

68. $\int \frac{x^3}{x^4+3x^2+2}dx$

Integrals of the form $\int \frac{p(x)}{ax^2+bx+c}dx$, where p(x) is a polynomial of degree two or more

To evaluate this type of integral we divide the numerator by denominator and express as $Q(x)+\frac{R(x)}{ax^2+bx+c}$ where R(x) is a linear function of x.

$$\therefore \int \frac{P(x)}{ax^2+bx+c}dx = \int Q(x)dx + \int \frac{R(x)}{ax^2+bx+c}dx$$

Evaluate the following integrals :

69. $\int \frac{x^3+x+1}{x^2-1}dx$

70. $\int \frac{(1-x^2)}{x(1-2x)}dx$

71. $\int \frac{x^2+1}{x^2-5x+6}dx$

72. $\int \frac{x^2(x^4+4)}{x^2+4}dx$

73. $\int \frac{x^2}{x^4 + 6x^2 + 12} dx$

Integrals of the form $\int \frac{P(x)}{\sqrt{ax^2 + bx + c}} dx$

Algorithm

Step1. Write the numerator px + q in the following form:

$$px + q = \lambda \left\{ \frac{d}{dx}\left(ax^2 + bx + c\right) \right\} + \mu \text{ i.e, } px + q = \lambda\left(2ax + b\right) + \mu$$

Step2. Obtain the value of λ and μ by equating the coefficient of like power of x on both sides.

Step3. Replace $px + q$ by λ (2ax + b) + μ in the given integral to get

$$\int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx = \lambda \int \frac{2ax + b}{\sqrt{ax^2 + bx + c}} dx + \mu \int \frac{1}{\sqrt{ax^2 + bx + c}} + c$$

Step4. Integrate RHS in step 3. And put λ and μ obtained in step 2.

Evaluate: $\int \frac{x + 2}{\sqrt{x^2 + 5x + 6}} dx$

74. $\int \sqrt{\frac{a - x}{a + x}} dx$

75. $\int \frac{x + 2}{\sqrt{x^2 - 1}} dx$

76. $\int \frac{2x + 1}{\sqrt{x^2 + 4x + 3}} dx$

77. $\int \frac{2x + 3}{\sqrt{x^2 + 4x + 5}} dx$

78. $\int \frac{5x + 3}{\sqrt{x^2 + 4x + 10}} dx$

79. $\int \frac{x + 2}{\sqrt{x^2 + 2x + 3}} dx$

Intergral of the form $\int \frac{1}{a\sin x + b\cos x + c} dx, \int \frac{1}{a + b\sin x} dx,$

$\int \frac{1}{a+b\cos x}dx, \int \frac{1}{a\sin x+b\cos x}dx,$

Algorithm

Step1. Put $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$ and $\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$ and simplify.

Step2. Replace $1+\tan^2 x/2 = \sec^2 x/2$

Step3. Put $\tan x/2 = t$ so that $1/2\sec^2 x/2dx = dt$ thus

$\int \frac{1}{at^2+bt+c}dt$

Evaluate:

80. $\int \frac{1}{1+\sin x+\cos x}dx$

81. $\int \frac{1}{1-2\sin x}dx$

82. $\int \frac{1}{5+4\cos x}dx$

83. $\int \frac{1}{3+2+\sin x+\cos x}dx$

84. $\int \frac{1}{\sin x+\sqrt{3}\cos x}dx$

85. $\int \frac{1}{1+\tan x}dx$

Integrals of the from $\int \frac{a\sin x+b\cos x+c}{p\sin x+q\cos x+r}dx$

To evaluate this type of integrals, we use the following algorithm.

Algorithm

Step1. Write a $a\sin x+b\cos x+c = \lambda(p\cos x-q\sin x)+v$

Step2. Obtain the value of λ and μ by equating the coefficient of sinx, cosx, and constant term on both sides.

Step3. Replace $\int \frac{a\sin x+b\cos x+c}{p\sin x+q\cos x+r} = \lambda \int \frac{p\cos x-q\sin x}{p\sin x+q\cos x+r}dx$

$+\mu\int \frac{p\sin x+q\cos x+r}{p\sin x+q\cos x+r}+V\int \frac{1}{p\sin x+q\cos x+r}dx+C$

Step4. Integrate RHS in step 2. And put ë and ì obtained in step 2.

86. Evaluate : $\int \frac{3\cos x + 2}{\sin x + 2\cos x + 3} dx$

87. Evaluate : $\int \frac{2\tan x + 3}{3\tan x + 4} dx$

Integration by parts

$$\int uv\, dx = u\left\{\int v\, dx\right\} - \int\left\{\frac{du}{dx}.\int v\, dx\right\}dx$$

Proper choice of first and second function-

ILATE where

I – Inverse trigonometric function

L – logarithmic function

A – algebraic function

T – Trigonometric function

E – exponential function

Practice Problems

Evaluate :

1) $\int x\sin 3x dx$ 2) $\int x\log x dx$

3) $\int x\sec^2 x dx$ 4) $\int x^2 e^x dx$

5) $\int x\sin^{-1} x dx$ 6) $\int x\tan^{-1} x dx$

7) $\int x\sec^{-1} x dx$ 8) $\int x\tan^{-1} x dx$

9) $\int (\sin^{-1} x)^2 dx$ 10) $\int x\sin^{-1} x dx$

11) $\int x^3 e^x dx$ 12) $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}} dx$

13) $\int \frac{\sqrt{x^2+1}(\log(x^2+1) - 2\log x)}{x^4} dx$

14) $\int xe^x dx$

15) $\int \frac{x\cos^{-1}x}{1+\cos x}dx$

16) $\int \frac{\log x}{(x+1)^2}dx$

17) $\int \frac{x\cos^{-1}x}{\sqrt{1-x^2}}$

18) $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

19) $\int (x+1)\log x dx$

20) $\int \frac{(x\tan^{-1}x)}{(1+x^2)^{3/2}}dx$

21) $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$

22) $\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}dx$

Integral of the form $\int e^x\{f(x)+f'(x)\}dx = e^x f(x)+c$

1. Evalute: $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$
2. Evalute: $\int e^x(\sin x+\cos x)dx$
3. Evalute: $\int e^x\left(\frac{2+\sin 2x}{1+\cos 2x}\right)dx$
4. Evalute: $\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$
5. Evalute: $\int e^x \frac{x}{(x+1)^2}dx$
6. Evalute: $\int e^x \frac{x-3}{(x-1)^3}dx$

7. Evalute : $\int \left\{ \log(\log x) + \frac{1}{(\log x)^2} \right\} dx$

8. Evalute : $\int e^{2x} \left(\frac{1+\sin 2x}{1+\cos 2x} \right) dx$

9. Evalute : $\int e^{x} \left(\frac{\sin 4x - 4}{1-\cos 4x} \right) dx$

10. Evalute : $\int \frac{2-x}{(1-x)^2} e^x dx$

11. Evalute : $\int e^x \frac{(x-4)}{(x-2)^3} e^x dx$

12. Evalute : $\int \frac{\sqrt{1-\sin x}}{1+\cos x} e^{-x/2} dx$

13. Evalute : $\int e^{2x} \left(\frac{1-\sin 2x}{1-\cos 2x} \right) dx$

Integrals of the from $\int e^{ax} \sin bx$ *and* $\int e^{ax} \cos bx$

in this section, we will discuss problems based upon integral of the from $\int e^{ax} \sin bx \; dx$ and $\int e^{ax} \cos bx \; dx$. In order to evaluate this type of integrals, we may use the following formulae:

$$\int e^{ax} \sin bx \; dx = \frac{e^{ax}}{a^2+b^2} (\text{acosbx+bsinbx})+\text{c}$$

1) Evaluate the following integrals :

2) $\int e^{ax} \cos bx \; dx$ 3) $\int e^{2x} \sin x dx.$

4) $\int e^{2x} \sin(3x+1) \; dx.$

Integrals of the form $\int \sqrt{ax^2+bx+c}$ and

$$\int (px+a)\sqrt{ax^2+bx+cdx}$$

1) $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$

2) $\int \sqrt{a^2+x^2}\,dx = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{1}{2}a^2 \log|x+\sqrt{a^2+x^2}| + c$

3) $\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2 \log|x+\sqrt{a^2-x^2}| + c$

Integrals of the from $\int \sqrt{ax^2+bx+c}\ dx$ In order to evaluate this type of integrals, we use the following algorithm.

Step 1 Make coefficient of x^2 as one by taking ‘a’ common to obtain

$$x^2+\frac{b}{x}x+\frac{c}{a}\ .$$

Step 2 : Add and subtract $\left(\frac{b}{2a}\right)^2$ in $x^2+\frac{b}{a}x+\frac{c}{a}$ to obtain.

$\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}$ After applying these two step the integral educes to one of the following three forms:

$$\int \sqrt{a^2+x^2}\,dx, = \int \sqrt{a^2-x^2}\,dx, \int \sqrt{x^2-a^2}\,dx\ .$$

Practice Problem

1)Evaluate $\int \sqrt{x^2+2x+5dx}$.

2)Evaluate $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\ dx$

3)Evaluate $\int \sqrt{3+2x-x^2}\,dx$

4)Evaluate $\int \sqrt{1+x-2x^2}\,dx$

5)Evaluate $\int \sqrt{16x^2+25}\ dx$

6)Evaluate $\int x^2\sqrt{a^6-x^6}\,dx$

7)Evaluate $\int \frac{\sqrt{16+(\log x)^2}}{x}\ dx$

Integrals of the from $\int (px+q)\sqrt{ax^2+bx+c}\ dx$ In order to evaluate this type of integrals, we use the following algorithm.

Step 1 Express $px+q$ as px+q=$\lambda\frac{d}{dx}\left(ax^2+bx+c\right)+\mu$ i.e.

px+q=λ(2ax+b)+μ

Step 2 Obtain the values of λ and μ by equating the coefficient of x and constant terms on both sides.

Step 3 Replace px+q by λ(2ax+b)+μ in the integral to obtain

$$=\int (px+q)\sqrt{ax^2+bx+cdx}=$$

$$\lambda\int(2ab+b)\sqrt{ax^2+bx+cdx}\ +\mu\int\ \sqrt{ax^2+bx+cdx}$$

Step 4 To evaluate first integral on RHS, use the following

$\int\left(f(x)\right)^n f'(x)dx=\int\frac{\{f(x)\}^{n+1}}{n+1}$ Evaluate second integral on RHS by the method discussed in the previous section.

(i) Evaluate $\int(3x-2)\sqrt{x^2+x+1}\ dx$

(ii) Evaluate $\int (x-3)\sqrt{x^2+3x-18}\ dx$

(iii) Evaluate $\int (x+3)\sqrt{3-4x-x^2}dx$

integration of rational algebraic functions by using partial fractions

partial fractions If f(x) and g(x) are two polynomials, then $\frac{f(x)}{g(x)}$ defines a rational algebraic function or rational function of x.

If degree of f(x) < degree of g(x), then $\frac{f(x)}{g(x)}$ is called a proper rational function.

If degree of f(x) > degree of g(x), then $\frac{f(x)}{g(x)}$ is called a improper rational function.

CASE I When denominator is expressible as the product of non-repeating liner factors.

Let $g(x)=(x-a_1)(x-a_2)..........(x-a_n)$. Then, we assume that

$$=\frac{f(x)}{g(x)}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+........+\frac{A_n}{x-a_n}$$

Practice Problems

1) Evaluate $\int \frac{2x-1}{(x-1)(x+2(x-3)}\ dx$

2) Evaluate $\int \frac{2x}{(x^2+1)(x^2+2)}dx$

3) Evaluate $\int \frac{1}{\sin x-\sin 2x}dx$

4) Evaluate $\int \frac{3x+1}{(x-2)^2(x+2)}dx$

5) Evaluate $\int \frac{x^2+1}{(x-1)^2(x+3)}dx$

6) Evaluate $\int \frac{8}{(x+2)(x^2+4)}dx$

7) Evaluate $\int \frac{x^2}{(x^2+1)\ (x^2+4)}dx$

8) Evaluate $\int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}dx$

9) Evaluate $\int \frac{1}{\sin x(2\cos^2 x-1)}dx$

10) Evaluate $\int \frac{5x}{(x+1)(x^2-4)}dx$

11) Evaluate $\int \frac{\sin 2x}{(1+\sin x)(2+\sin x)}dx$

12) Evaluate $\int \frac{2x}{(x^2+1)(x^2+3)}dx$

13) Evaluate $\int \frac{1}{x\log x(2+\log x)}dx$

14) Evaluate $\int \frac{x^2+x+1}{(x^2+1)(x+2)}dx$

15) Evaluate $\int \frac{x^2}{(x^2+4)(x^2+9)}dx$

16) Evaluate $\int \frac{x}{(x^2+1)(x+1)}dx$

17) Evaluate $\int \frac{x}{(x-1)^2(x+2)}dx$

18) Evaluate $\int \frac{3}{(1-x)(1+x^2)}dx$

19) Evaluate $\int \frac{2x^2+1}{x^2(x^2+4)}dx$

20) Evaluate $\int \frac{x}{(x^2+1)(x+1)}dx$

21) Evaluate $\int \frac{2x+1}{(x-2)(x-3)}dx$

22) Evaluate $\int \frac{1}{x(x^4-1)}dx$

23) Evaluate $\int \frac{1}{x(x^n+1)}dx$

24) Evaluate $\int \frac{x^2+1}{(x^2+4)(x^2+25)}dx$

25) Evaluate $\int \frac{3x-2}{(x^2+4)(x^2+25)}dx$

26) Evaluate $\int \frac{18}{(x+2)(x^2+4)}dx$

27) Evaluate $\int \frac{x}{(x+2)(x^2+1)}dx$

28) Evaluate $\int \frac{3x+5}{x^3-x^2-x+1}dx$

29) Evaluate $\int \frac{x^2+x+1}{(x+1)^2(x+2)}dx$

30) Evaluate $\int \frac{1}{x(x^3+8)}dx$

31) Evaluate $\int \frac{\cos x}{(1-\sin x)(2+\sin x)}dx$

32) Evaluate $\int \frac{1}{(x^2+1)(x^2+2)}dx$

33) Evaluate $\int \frac{1}{x^4-1}dx$

34) Evaluate $\int \frac{1}{\sin x+\sin 2x}dx$

integrals of the from $\int \frac{x^2+1}{x^4+\lambda x^2+1}dx,\ \int \frac{x^2-1}{x^4+\lambda x^2+1}dx,$

$\int \frac{1}{x^4+\lambda x^2+1}dx,$ **where** $\lambda \in R$

To evaluate this type of integrals, we use the following algorithm.

Algorithm

Step 1 Divide numerator and denominator by x^2.

Step 2 Express the denominator of integrand in the form $\left(x+\frac{1}{x}\right)^2 \pm k^2$

Step 3 Introduced $d\left(x+\frac{1}{x}\right)$ or, $d\left(x-\frac{1}{x}\right)$ or both in the numerator.

Step 4 Substitute $x+\frac{1}{x}=t$ or, $x-\frac{1}{x}=t$ as the case may be.

This substitute reduces the integral in one of the following form

$$\int \frac{1}{x^2+a^2}dx \qquad \int \frac{1}{x^2-a^2}dx\ .$$

Step 5 Use the appropriate formula.

Practice Problems

1)Evaluate $\int \frac{x^2+1}{x^4+1}dx$

2)Evaluate $\int \frac{x^2+4}{x^4+16}dx$

3)Evaluate $\int \frac{1}{x^4+1}dx$

4)Evaluate $\int \left\{\sqrt{\tan\theta}+\sqrt{\cot\theta}\right\}d\theta$.

5)Evaluate $\int \frac{1}{\sin^4 x+\cos^4 x}dx$

6)Evaluate $\int \frac{x^2-1}{x^4+1}dx$

7)Evaluate $\int \frac{1}{\sin^4 x+\sin^2 x\cos^2 x+\cos^4 x}dx$

Integrals of the from $\int \frac{\phi(x)}{p\sqrt{Q}}dx,$ where p and q both are linear functions of x

To evaluate this type of integrals we put Q = t^2 i.e., To evaluate integrals of the form $\int \frac{1}{(ax+b)\sqrt{cx+d}}dx,$ · , put cx+d=t^2

Practice Problems

Following examples illustrate the procedure.

1)Evaluate $\int \frac{1}{(x-3)\sqrt{x+1}}dx.$

2)Evaluate $\int \frac{1}{(x^2-4)\sqrt{x+1}}dx.$

3)Evaluate $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}dx.$

4)Integrals of the from $\int \frac{\phi(x)}{p\sqrt{Q}}dx,$, where p is a linear expression and q is a quadratic expression

5)To evaluate this type of integrals we put $P = 1/t$ i.e , To evaluate integrals of the form $\int \frac{1}{(ax+b)\sqrt{px^2+qx+r}}dx$ we put $ax+b=\frac{1}{t}.$

6)Following examples will illustrate the procedure.

7)Evaluate $\int \frac{1}{(x+1)\sqrt{x^2-1}}dx$

Integrals of the from $\int \frac{\phi(x)}{p\sqrt{Q}}dx,$, where p is q both are pure quadratic expression in x i.e. $P = ax^2+b$ and $Q=cx^2+d$

To evaluate this type of integrals we put $x=\frac{1}{t}$ and then $c+dt^2=u^2$

i.e., To evaluate integrals of the form $\int \frac{1}{(ax^2+b)\sqrt{cx^2+d}}dx$, we put $x=\frac{1}{t}$ o obtain $\int \frac{-tdt}{\left(a+dt^2\right)\sqrt{c+dt^2}}dx$ and then substitute

$c+dt^2=u^2\cdot$

Following examples will illustrate the procedure.

Practice Problems

1) Evaluate $\int \frac{1}{x^2\sqrt{1+x^2}}dx.$

2) Evaluate $\int \frac{1}{x\sqrt{ax-x^2}}dx.$

3) Evaluate $\int \frac{1}{(x-1)\sqrt{x+2}}dx.$

4) Evaluate $\int \frac{1}{(x+1)\sqrt{x^2+x+1}}dx.$

5) Evaluate $\int \frac{1}{(1+x)^2\sqrt{1-x^2}}dx.$

6) Evaluate $\int \frac{x^2}{(x-1)\sqrt{x+2}}dx.$

7) Evaluate $\int \frac{x^2}{1+x^3}dx.$

8) Evaluate $\int \frac{x^2+4x}{x^3+6x+5}dx.$

9) Evaluate $\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}dx.$

10) Evaluate $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx.$

11)Evaluate $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx.$

12) Evaluate $\int \frac{(1+\log x)^2}{x} dx.$

13) Evaluate $\int \sec^2(7-4x)dx$

14) Evaluate $\int \frac{\log x}{x} dx.$

15) Evaluate $\int 2^x dx$

16) Evaluate $\int \frac{1-\sin x}{\cos^2 x} dx.$

17) Evaluate $\int \frac{x^3-1}{x^2} dx.$

18) Evaluate $\int \frac{x^3-x^2+x-1}{x-1} dx.$

19) Evaluate $\int \frac{e^{\tan^{-1}x}}{1+x^2} dx.$

20) Evaluate $\int \frac{1}{\sqrt{1-x^2}} dx.$

21) Evaluate $\int \sec x(\sec x + \tan x)dx.$

22) Evaluate $\int \frac{1}{x^2+16} dx.$

23) Evaluate $\int (1-x)\sqrt{x} \; dx.$

24) Evaluate $\int \frac{x+\cos 6x}{3x^2+\sin 6x} dx.$

25) Evaluate $\int\left(\frac{x-1}{x^2}\right)e^x\ dx=\int(x)e^x+c\ dx.$ then write the value of f (x).

26) If $\int e^x(\tan x+1)\ \sec x dx=e^x f(x)+c$ then write the value f (x).

27) Evaluate $\int\frac{2}{1-\cos 2x}dx.$

28) Write the anti derivative of $\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)$.

29) Evaluate $\int\cos^{-1}(\sin x)dx$

30) Evaluate $\int\frac{1}{\sin^2 x\cos^2 x}$

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17 Definite Integrals

Fundamental throrem of integrel calculus

Statement:

let $\phi(x)$ be theprimitive or antiderivative of a continuous function f(x0 defined on [a,b]

i.e. $\frac{d}{dx}\{\phi(x)\} = f(x)$. Then the definite Integral of f(x) over [a,b] is denoted by $f(x)d(x) = \phi(b) - \phi(a)$

Evaluation of definite integral

Algorithm:

STEP 1. Find the definite integral $f(x)d(x)$. let this be$\phi(x)$.

STEP 2. Evaluate $\phi(b)\ \ \phi(a)$.

A. Evaluate:

1) $\int_{0}^{\pi/4} \sqrt{1-\sin 2xdx}$ [Ans. 1]

2) $\int_{0}^{1} xe^{x} dx$ [Ans. 1]

3) $\int_{0}^{1} \left\{ xe^{x} + \sin\frac{\pi x}{4} \right\} dx$ $\left[Ans: 1 + \frac{4}{\pi}\ \frac{2\sqrt{2}}{\pi} \right]$

4) $\int_{0}^{1} x\log(1+2x)dx$ $\left[Ans.: 3/8\log 3 \right]$

5) $\int_{1}^{2} \frac{5x^2}{x^2+4x+3} dx$ $\left[Ans.: 5 - 10\log(15/18) + 25/2\log(6/5) \right]$

6) $\int_{\pi/6}^{\pi/4} \cos ec\, dx$ $\left[Ans.: \log(\sqrt{2}-1)-\log 2-\sqrt{3}\right]$

7) $\int_{0}^{\pi/2} \cos^2 x\, dx$ $\left[Ans.: \frac{\pi}{4}\right]$

8) $\int_{0}^{\pi/2} x^2 \sin x\, dx$ $\left[Ans.: \pi - 2\right]$

9) $\int_{\pi/4}^{\pi/2} \cos 2x \log \sin x\, dx$ $\left[Ans.: 1/4 \log 2 \frac{\pi}{8} + 1/4\right]$

10) $\int_{0}^{1} \frac{2x+3}{5x+1} dx$ $\left[Ans.: (1/5)\log 6 + \frac{3}{\sqrt{5}} \tan^{-1}\sqrt{5}\right]$

11) $\int_{-1}^{1} \frac{1}{x^2+2x+5} dx$ $\left[Ans.: \frac{\pi}{8}\right]$

12) $\int_{\pi/2}^{\pi} e^2 \left(\frac{1-\sin x}{1-\cos x}\right) dx$ $\left[Ans.: e^{\frac{\pi}{2}}\right]$

13) $\int_{1}^{2} \left(\frac{x-1}{x^2}\right) e^x dx$ $\left[Ans.: e^2 - e\right]$

14) $\int_{1}^{2} \frac{1}{\sqrt{1+x}-\sqrt{x}} dx$

15) $\int_{1}^{2} \frac{x}{(x+1)(x+2)}$ dx

16) $\int_{1}^{\pi/2} \sin^3 x dx$

17) $\int_{0}^{\pi} \left(\sin^2 - \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx$

18) $\int_{1}^{2} e^{2x} \left(\frac{1}{x} - \frac{1}{2x^2} \right) dx$

19) $\int_{1}^{2} \frac{1}{\sqrt{(x-1)(2-x)}} dx$

20) $\int_{0}^{\pi} (\sin 2x - \cos 2x) dx$

Evaluation of definite integral by substitution:

Algorithm:

Step1. The definite integral in the form I = $\int_{a}^{b} f(x) g'(x) dx$

Step2. Put $t = g(x)$ which gives $g'(x) dx = dt$

Step3. Put $x = a$ to get $g(a)$ & $x = b$ To get g(b).

Step4. Put $g(x)dx = dt$ dt hence $i = \int_{g(a)}^{(b)} f(t) dt$

1. $\int_{0}^{1} \sin^{-1} x \, dx$

2 $\int_{0}^{\pi/2} \sqrt{\cos o} \sin^3 \theta \, d\theta$

3. $\int_0^{1/2} \frac{1}{(1+x^2)\sqrt{1-x^2}}\,dx$

4. $\int_0^{\pi/4} \frac{1}{\cos^3 x\sqrt{2\sin 2x}}\,dx$

5. $\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{\left(1-x^2\right)^{3/2}}\,dx$

6. $\int_0^{1} \sin^{-1}\left(\frac{2x}{1-x^2}\right)dx$

7. $\int_0^{\pi/4} \tan^3 x\,dx$

8. $\int_0^{\pi/2} \left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx$

9. $\int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x+\cos^4 x}\,dx$

10. $\int_0^{\pi/4} \frac{\sin 2x}{\sin^4 x+\cos^4 x}\,dx$

11. $\int_0^{\pi/2} \frac{\cos^2 x}{\cos^2 x+4\sin^2 x}\,dx$

12. $\int_0^{\pi/2} \frac{1}{(a^2 \cos^2 x + b^2 \sin^2 x)^2}$

13. $\int_0^1 x\left(\tan^{-1} x\right)^2 dx$

14. $\int_2^4 \frac{x}{x^2+1} dx$

15. $\int_0^2 \sqrt{x+2} dx.x$

16. $\int_0^{\pi/4} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$

17. $\int_0^{\pi/2} \frac{x+\sin x}{1+\cos x} dx$

18. $\int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$

19. $\int_0^{\pi/4} \frac{\sin x + \cos x}{3+\cos x} dx$

20. $\int_0^1 \sqrt{\frac{1-x}{1+x}} dx$

21. $\int_{-1}^1 5x^4 \sqrt{x^5 + idx}$

22. $\int_{0}^{\pi/2} \frac{\cos^2 x}{1+3\sin^2 x} dx$

23. $\int_{0}^{\pi/4} \sin^3 2t \cos 2t dt$

24. $\int_{4}^{9} \frac{\sqrt{x}}{\left(30-x^{3/2}\right)^2} dx$

25. $\int_{0}^{\pi} \sin^3 x(1+2\cos x)(1+\cos x)^2 dx$

26. $\int_{0}^{\pi/2} 2\sin x \cos x \tan^{-1}(\sin x) dx$

Properties of definite integrals

Property i : $\int_{a}^{b} f(x)dx = \int_{a}^{b} f(t)dx$

Property ii : (order of integression) $\int_{a}^{b} f(x)dx = \int_{a}^{b} f(x)dx$

Property iii : (additivity) $\int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b}(x)dx,$.

where a < c < b

27. $\int_{-1}^{2} \left|x^3 - x\right| dx$

28. $\int_{1}^{4}\left(|x-1|+|x-2|+|x-3|\right)dx$

29. $\int_{-\pi/2}^{\pi/2}\left(\sin|x|+\cos|x|\right)dx$

30. $\int_{0}^{4}|x-1|\,dx$

31. $\int_{1}^{4}\left\{|x-1|+|x-2|+|x-4|\right\}dx$

32. $\int_{-5}^{0} f(x)\,dx, where\ f(x)=|x|+|x+2|+|x+5|$

33. $\int_{0}^{4}\left(|x|+|x-2|+|x-4|\right)dx$

34. $\int_{-1}^{2}\left(|x+1|+|x|+|x-1|\right)dx$

35. $\int_{0}^{2\pi}\cos^{-1}\left(\cos x\right)dx$

36. $\int_{0}^{\sqrt{3}}\tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$

Property iv : If f(x) is a continuous function defined on [a,b], then

$$\int_{a}^{b} f(x)\,dx=\int_{a}^{b} f(a+b-c)\,dx$$

37. $\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}} dx$

38. $\int_{0}^{a} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}} dx$

39. $\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\tan x}}$

40. $\int_{-\pi/2}^{\pi/2} \frac{x \sin x}{e^{x}+1} dx$

41. $\int_{0}^{2\pi} \log(\sec x + \tan x) dx$

Property v : IF f(x) is defined on [0,a], then

$$\int_{0}^{a} f(x) dx = \int_{0}^{a} f(a-x) dx$$

42. $\int_{0}^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4}$

43. $\int_{0}^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$

44. $\int_{0}^{\pi/2} \log \tan x \, dx$

45. $\int_{0}^{\pi/4} \log(1+\tan x)dx$

46. $\int_{0}^{\pi/2} \frac{\sin x - \cos x}{1+\sin x \cos x}$

47. $\int_{0}^{\pi/2} (2\log\sin - x\log\sin 2x)dx$

48. $\int_{0}^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}dx$

49. Prove that $\int_{0}^{2a} f(x)dx = \int_{0}^{2a} f(2a-x)dx$

50. Evalute $\int_{0}^{1} x(1-x)^x dx$

51. $\int_{0}^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}$

52. $\int_{0}^{1} \cot^{-1}(1-x+x^2)dx$

53. If f and g continuous o [0,a] and satisfy $f(x) = f(x-a)g(x) + g(a-x) = 2$ show that

$$\int_{0}^{a} f(x)g(x)dx = \int_{0}^{a} f(x)dx$$

Property vi : IF f(x) is defined on [0,2a], then

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx = \int_0^a f(x) + f(2a-a)dx$$

Property vii : IF f(x) is defined on [0,a], such that f(a-x) = f(x). then

$$\int_{-a}^{a} f(x)dx = \int_0^a f(x) + f(-x)dx$$

54. $\int_0^{2\pi} \frac{1}{1+e^{\sin x}} dx$

55. $\int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2 \sin^2 x} dx$

56. $\int_0^{\pi} \frac{x\sin x}{1\cos^2 x} dx$

57. $\int_0^{\pi} \frac{x}{1+\sin x} dx$

58. $\int_0^{\pi} \frac{x\tan x}{\sec x + \tan x} dx$

59. $\int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x} dx$

Property viii : If f(x) is defined on [-a,a], then

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f\left(x\right)dx, \quad if \;\; f\left(x\right)$$

60. $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^{x}} dx$

Property ix : If f(x) is defined on [-a,a], then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

if f(x) is even $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x) + f(-x)\ dx$

Evaluate :

61. $\int_{-\pi/2}^{\pi/2} \sin^7 x\,dx$

62. ` $\int_{-\pi/2}^{\pi/2} \sin^2 x\,dx$

63. $\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x\,dx$

64. $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}} dx$

65. $\int_{-1}^{1} \frac{x^3+|x|+1}{x^2+2|x|+1}$

66. $\int_{-\pi}^{\pi} (\cos ax - \sin bx)^2\, dx$

67. $\int_{-1}^{3/2} |x \sin \pi x| \, dx$

Property x: if f(x) is continuous function defined on [0,2a], the*n*

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)dx & if \quad f(2a-x) = f(x) \\ 0 & f(2a-x) = -f(x) \end{cases}$$

Evaluate:

68. $\int_0^{2\pi} \cos^5 x \, dx$

69. $\int_0^{2\pi} \log \sin x dx = \int_0^{2\pi} \log \cos x dx = -\frac{\pi}{2} \log 2$

70. $\int_0^{\pi} \log(1 + \cos x) \, dx$

71. $\int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x) \, dx$

Practice Problems:

72. $\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}} dx$

73. $\int_0^{\pi/2} \frac{\sin^{3/2}}{\sin^{3/2} x + \cos^{3/2} x} dx$

74. $\int_0^1 \frac{\log(1+x)}{1+x^2}dx$

75. $\int_0^\pi \frac{x\tan x}{\sec x \cos ecx}dx$

76. . $\int_0^\pi \frac{x\tan x}{\sec x \cos ecx}dx$

77. $\int_0^\pi \frac{x\sin x}{1+\sin x}dx$

78. $\int_0^\pi x\sin x\cos^2 x \_ dx$

79. $\int_0^2 x\sqrt{2-xdx}$

80. $\int_0^1 \log\left(\frac{1}{x}-1\right)dx$

81. $\int_{-1}^1 |x\cos\pi x|\,dx$

B. Integration as the limit of a sum

Let f(x) be a continuous real valued function defined on the closed interval [a,b] which is divided into n equal parts each of width h then , $h = \frac{b-a}{n}$

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a+h) + F(a+2h) +$$

$$f(a+3h) ....+ f(a+nh)]$$

Following result will be helpful in evaluating definite integral as limit of sums:

1. $1+2+3+.....+(n-1)=\dfrac{n(n-1)}{2}$

2. $1^2+2^2+3^2+........................+(n+1)^2=\dfrac{n(n-1)(2n-1)}{6}$

3. $1^3+2^3+3^3+........................+(n+1)^3=\left\{\dfrac{n(n-1)^2}{2}\right\}$

4. $a+ar+ar^2+ar^3+...............................+ar^{n-1}=a\left(\dfrac{r^n-1}{r-1}\right), r\neq 1$

5. $\sin a+\sin(a+h)+\sin(a+2h)+.............+\sin(a+(n-1)h)$

$$=\frac{\sin\left\{a+\left(\frac{n-1}{2}\right)h\right\}\sin(nh/2)}{\sin(h/2)}$$

6.

$\cos a+\cos(a+h)+\cos(a+2h)+...................+\cos(a+(n-1)h)$

$$=\frac{\cos\left\{a+\left(\frac{n-1}{2}\right)h\right\}\sin(nh/2)}{\sin(h/2)}$$

Evaluate the following integrals as limit of sums:

1. $\int_1^3 \left(2x^2+5\right)dx$

2. $\int_1^3 \left(x^2+x\right)dx$

3. $\int_1^3 \left(x^2+5x\right)dx^{\int_{-1}^{1} e^x dx}$

4. $\int_0^1 e^{2-3x}dx$

5. $\int_1^3 \left(e^{2-3x}+x^2+1\right)dx$

6. $\int_0^2 e^x dx$

7. $\int_0^2 \left(x^2+2\right)dx$

8. $\int_0^4 \left(x+e^{2x}\right)$

9. $\int_0^2 \left(x^2+x\right)dx$

10. $\int_0^2 \left(x^2+2x+1\right)dx$

11. $\int_0^2 \left(x^2 + 2x + 1\right) dx$

12. $\int_0^3 \left(2x^2 + 3x + 5\right) dx$

13. $\int_a^b x\, dx$

14. $\int_2^3 x^2 dx$

15. $\int_0^2 \left(x^2 - x\right) dx$

Evaluate each of the following integrals: (very short answer questions)

16. $\int_0^1 \frac{1}{1+x^2} dx$

17. $\int_2^3 \frac{1}{x} dx$

18. $\int_0^1 \frac{2x}{1+x^2} dx$

19. $\int_0^{\pi/4} \sin 2x\, dx$

20. $\int_0^{\pi/4} \tan x\, dx$

21. $\int_0^2 \sqrt{4 - x^2}\, dx$

22. $\int_0^1 x e^{x^2} dx$

23. $\int_e^{e^2} \frac{1}{x \log x} dx$

24. $\int_{e}^{\pi/2} e^{x}\left(\sin x-\cos x\right)dx$

25. $\int_{2}^{4}\frac{x}{x^{2}+1}dx$

26. If $\int_{0}^{1}\left(3x^{2}+2x+k\right)dx=0$ find the value of k

27. If $\int_{0}^{a}3x^{2}dx=8$ write the value of a

28. If $f\left(x\right)=\int_{0}^{x}t\sin t\,dt$ then write the value of $f'(x)$

29. If $\int_{0}^{a}\frac{1}{4+x^{2}}dx\frac{\pi}{8}$ find the value of a

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18 Areas of Bounded Regions

Area as a definite integral:

Theorem:

let f(x) be a continuous function defined on [a,b] . Then the area bounded by the curve y = f(x) , the x-axis and the ordinates x=a and x=b is given by

$$\int_a^b |f(x)|\, dx \quad \text{or} \quad \int_a^b |y|\, dx$$

Area using vertical strips

Area of region bounded by the curve y = f(x) , x axis and the ordinate x=a and x=b we use the following algorithm.

ALGORITHM:

Step1. Sketch the curve and identify the bounded regionwhose area is to be found.

Step2. Take an arbitrary point P(x,y) on the curve and construct a representative strip of width dx having two ends on x axis at points $\left(x-\frac{dx}{2},0\right)$ and $\left(x+\frac{dx}{2},0\right)$ and (x,0) as the mid-point of its base.

Step3. Construct an approximate rectangle whose base is same as that of the representative strip and height is $|y| = |f(x)|$.

Step 4. Find the area of approximating rectangle as $|y| = |f(x)|\, dx$.

Step5. Find the value of x, say x=a and x=b form the integral $\int_a^b |f(x)|\,dx$ or $\int_a^b |y|\,dx$ so obtained in the required area.

Practice problems

1) Find the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.
2) Find the area of the region bounded by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
3) Find the area of the region bounded by the line $y = 3x + 2$, the x-axis and the ordinates x=-1 and x=1.
4) Using integration, find the area of the region bounded by the following curves, after making a rough sketch: $y = 1 + |x+1|$, $x = -3, x = 3, y = 0$.
5) Using integration find area of the triangle formed by positive x-axis and tangent and normal to the circle $x^2 + y^2 = 4at\left(1, \sqrt{3}\right)$.
6) Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$
7) Using integration, find the area of the regions bounded by the line y-1=x, the x-axis and the ordinates x =-2 and x=3.
8) Draw a rough sketch of the graph of the curve $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and evvluate the area of the region under the curve and above the x-axis.
9) Determine the area under the curve $y\sqrt{a^2 + x^2}$ included between the lines x=0 and x=a.

10) Sketch the graph $y = |x-5|$. Evaluate $\int_0^1 |x-5|\,dx$. What does this value of the integral represent on the graph.

11) Sketch the graph $y = |x+3|$ Evaluate $\int_{-6}^{0} |x+3a|ds$. What does this integral represent on the graph?

12) Find the area bounded by the curve $y = \cos x, x - axis$ and the ordinates x=0 and $x = 2\pi$.

13) Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and ordinates $x = ae$ and $x = 0$, where $= b^2 = a^2\left(1-e^2\right)$ and $e < 1$.

14) Find the area of the region bounded by the curve $x = at^2$, $y = 2at$ between the ordinates corresponding $t = 1 \; and \, t = 2$.

15) Find the area enclosed by the curve $x = 3\cos t$, $y = 2\sin t$.

Area using horizontal strips

The area of the regions, bounded by the curve $x = f(y)$, y axis and the lines y = c and y = d , we may use the following algorithm.

Algorithm

Step1. Sketch the curve and identify the bounded region whose area is to be found.

Step2. Take an arbitrary point P(x,y) on the curve and construct a representative strip of width dy and (y,0) as the mid-point of its base.

Step3. Construct an approximate rectangle whose base is same as that of the representative strip and height is $|x| = |f(y)|$

Step 4. Find the area of approximating rectangle as $|x| = |f(y)|dx$.

Step5. Find the value of x, say x = c and x = d form the integral

$\int_a^b |f(x) - g(x)|dx$ so obtained in the required Area

Practice problems

1. Find the area of the region bounded by the curve $y^2 = 2y - x$ and the Y axis. [ans.4/3 sq.unit]
2. Sketch the region lying in the 1st quadrant and bounded by $y = 9x^2$, $x = 0, y = 1$ and $y = 4$. Find the area of region. [ans.14/9 sq.units]
3. Find the area bounded by the curve $y^2 = 4ax$ and the lines y = 2a and y-axis. [ans.$2a^2/3$ sq.unit]
4. Find the area bounded by $y = -1$, $y- = 2$ $x = y^3$ and $x = 0$. [ans. 17/4 sq.unit]

Area between two curves by using vertical strips

Algorithm

Step 1. Draw given curves y = f(x) and y = g(x) and vertical lines x = a & x=b.

Step 2. Identify the region between curves and vertical lines in step 1.

Step 3. Take an arbitrary point P(x,y) on one of the curve say y = f(x) and draw a vertical line through P to meet the other curves y=g(x) at Q(x^2,y^2). Clearly y_1=f(x) & y_2= g(x).

Step 4. Draw a vertical approximately rectangle of width dx, height = $|y_1 - y_2| = |f(x) - g(x)|$ such that P(x,y_1) & Q(x,y_2) are midpoint.

Step 5. Find the area $\Delta A = \left|f(x) - g(x)\right| dx$ Then A= $\int_a^b \left|f(x) - g(x)\right|$

to find the area of the region between y = f(x) , y = g(x) and lines x=a & x=b. hence

$$A = \int_a^b \left|f(x) - g(x)\right| dx = \int_a^c \left|f(x) - g(x)\right| dx + \int_a^b \left|f(x) - g(x)\right| dx$$

16) Find the area of the region included between the parabola $y = \frac{3x^2}{4}$ and the line 3x – 2y + 12 = 0. [ans.A=27sq.unit]

17) Find the area bounded by the curve $x^2 = 4y$ and the straight line = x = 4y -2. [ans.9/8sq.unit]

18) Find the area of the region enclosed by the parabola $y^2 = 4ax$ and the line y = mx. [ans.$8a^2/3m^3$sq.unit]

19) Find the area of the region included between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ where a>0. [ans.$16a^2/3$ sq.unit]

20) Find the area of the region $\left\{(x, y) : x^2 \leq y \leq x\right\}$ [ans.1/6 sq.unit]

21) Find the area of the region $\left\{(x, y) : x^2 \leq y \leq |x|\right\}$ [ans.1/3 sq.unit]

22) Find the area of region bounded by the curve $y = x^3$ and the line y = x + 6 and x = 0. [ans.10 sq.unit]

23) Find the area of region $\left\{(x, y) : 0 \leq y \leq x^2 + 1, 0 \leq y \leq x + 1, 0 \leq x \leq 2\right\}$ [ans.23/6sq.unit]

24) Find the area of the region $\left\{(x, y) : x^2 + y^2 \leq 1 \leq x + y\right\}$ [ans.1/2 sq. unit]

25) Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the straight line $\frac{x}{a}+\frac{y}{b}=1$. ans.

$\left[1/2\left(\pi/2-1\right)ab\ sq.unit\right]$

26) Find the area of region $\left\{(x,y): y^2 \le 4x, 4x^2+4y^2 \le 9\right\}$.

$\left[ans.\ \frac{\sqrt{2}}{6}+\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}(1/3)s.q.unit\right]$

Or

Find the area of the circle $4x^2+4y^2=9$ which is interior to the parabola $y^2=4x$.

27) Find the area of the region $\left\{(x,y): x^2+y^2 \le 2ax, y^2 \ge ax, x \ge 0, y \ge 0\right\}$.

$\left[ans.\left(\pi/4-2/3\right)a^2 sq.unit\right]$

28) Find the area of the region enclosed between the two circles $x^2+y^2=1$ and $\left(x+1\right)^2+y^2=1$.

$\left[ans.\ \left(2\pi-\sqrt{3}/2\right)sq.unit\right]$

29) Using integration, find the area of triangle ABC whose vertices have coordinates A (2,5), B (4,7) and C(6,2). [ans.7 sq.unit]

30) Compute the area bounded by the lines x + 2y =2, y –x = 1 and 2x + y = 7. [Ans. 6 sq.unit]

31) Prove that the curves $y^2=4x$ and $x^2=4y$ divide the area of square bounded by $x=0, y=0, x=4, y=4$ into three equal parts. [Ans.16/3 sq.unit]

32) Calculate the area of the region bounded by the parabolas = 6x and = 6y. [ans.12 sq.unit]

33) Using integration, find the area of the region bounded by the triangle ABC whose vertices A, B, C are (-1,1), (0, 5) and (3, 2) respectively. [Ans.15/2 sq.unit]

34) Find the area of the region between the circles

$x^2 + y^2 = 4(x-2)^2 + y^2 = 4$ [ans. $8\pi/3 - 2\sqrt{3}$]

35) Prove that the area in the first quadrant enclosed by the x-axis, the line $x = \sqrt{3y}$ and the circle $x^2 + y^2 = 4\ is \pi/3$.

36) Find the area of the region bounded by $y = \sqrt{x}, \quad x = 2y + 3$ in the first quadrant and x-axis. [Ans.9 sq.unit]

37) Find the area common to the circle $x^2 + y^2 = 16a^2$ and the parabola $y^2 = 6ax$ ax. [Ans. $4a^2\left(4+\sqrt{3}\right)$ sq. unit]

Find the area of the region $\{(x, y) : y^2 \le 6ax\}$ and $\{(x, y) : x^2 + y^2 \le 16a^2\}$.

38) Find the area, lying above x-axis and included between the circle $x^2 + y^2 = 8x$ and the parabola $y^2 = 4x$

[Ans. $(4\pi - 32/3)\, sq.unit$]

39) Using integration, find the area of the region bounded by the triangle whose vertices are (-1, 2), (1 , 5) and (3, 4)

[Ans. 14.sq.unit]

40) Find the area of the region bounded by $y\sqrt{x}$ and $y = x$.

[Ans.1/6 sq.unit]

41) Find the area of the region in the first quadrant enclosed by x-

axis, the line $y = \sqrt{3}\,x$ and the circle $x^2 + y^2 = 16$.

[Ans. $8\pi / 3$ sq.unit]

42) Find the area enclosed by the curve $y = -x^2$ and the straight line x + y + 2 = 0. [Ans.9/2 sq.unit]

43) Using the method of integration, find the area of the region bounded by the following lines: 3x –y - 3 = 0, 2x + y – 12 = 0, x – 2y – 1 = 0. [Ans.11 sq.unit]

44) Using integration find the area of the triangle ABC coordinates of whose vertices are A(4, 1), B(6, 6) and C(8, 4).

[Ans. 7.sq.unit]

45) Using integration find the area of the region:

$$\left\{(x, y) : |x-1| \le y \le \sqrt{5-x^2}\right\}$$

$$\left(ans. \left\{ \frac{5}{2}\left(\frac{\sin^{-1} 2}{\sqrt{5}} + \frac{\sin^{-1} 1}{\sqrt{5}} \right) - \frac{3}{2} \right\} sq.unit \right)$$

46) Find the area of the region in the first quadrant enclosed by the x – axis, the line y=x and the circle $x^2 + y^2 = 32$.

[Ans. 4π *sq.unit*]

47) Find the area of the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$. [Ans.9/2 sq.unit]

48) Find the area of the region bounded by the curve $y = \sqrt{1+x^2}$, line y =x and the positive x- axis. [Ans. $\pi / 8$ sq.unit]

49) Find the area bounded by the lines $y = 4x + 5$, $y = 5 - x$ and $4y = x + 5$. [Ans. 15/2 sq.unit]

50) Find the area of the region enclosed between the two curves

$x^2 + y^2 = 9$ and $(x-3)^2 + y^2 = 9$ [Ans. $6\pi - 9\sqrt{3}/2$ sq.unit]

51) Find the area of the region . $\{(x, y) : x^2 + y^2 \le 4, x + y \ge 2\}$

[Ans. $(\pi - 2)$ sq.unit]

52) Using integration, find the area of the following region: $\left\{(x, y) : \frac{x^2}{9} + \frac{y^2}{4} \le 1 \le \frac{x}{3} + \frac{y}{2}\right\}$.

[Ans. $(3\pi / 2 - 2))$ sq.unit]

Areas between two curves by using horizontal strips

53) Find the area of the region bounded by the parabola = 2x and straight line x – y =4. [Ans. 18 sq.unit]

54) Find the area of the region bounded by the curve y = and the lines y = x + 6 and y = 0. [Ans. 28 sq.unit]

55) Find the area of the region bounded by the parabola =2x and straight line x – y =4. [Ans. 18 sq.unit]

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19 | Differential Equation

Defferential Equation:

An equations containing an independent variable , dependent variable and differential co-efficient of dependent variable w.r.t. independent variable is called a differential equation.

For instance,

(i) $\frac{dy}{dx} = 2xy$ (ii) $\frac{d^2y}{dx^2} - 5\left(\frac{dy}{dx}\right) + 6y = x^2$

(iii) $\frac{dy}{dx} = \sin x + \cos x$ (iv) $\left(\frac{d^3y}{dx^3}\right)^2 + \left(1 + \frac{dy}{dx}\right)^3 = 0$

Order of differential equation: The order of a differential is the order of the highest order derivative appearing in the examination.

Example: In the equation $\frac{d^2y}{dx^2} - 5\left(\frac{dy}{dx}\right) + 6y = x^2$ the order of highest derivative is 2 so,the order of differential equation is 2.

Degree of a differential equation: The degree of differential equation is the degree of the highest order derivative , when differential coefficients are made free from radicals and fractions.

Example: In the equation $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\frac{d^2y}{dx^2}$ the power of highest derivative is 2.so the degree of differential equation is 2.

Linear and non-linear differential equation:

A differential equation is a linear differential equation if it is expressed in the form $p_o\frac{d^ny}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + p_2\frac{d^{n-2}y}{dx^{n-2}} + ...p_{n-1}\frac{dy}{dx} + p_ny = Q$

Where $p_o, p_1, p_2, \ldots\ldots\ldots p_{n-1}, p_n$ and Q are either constants or function of independent variable.

Find the degree and order of differential equation:

1) $\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right) - 6y = \log x$

2) $\left(\frac{dy}{dx}\right) - 4\left(\frac{dy}{dx}\right)^2 + 7y = \sin x$

3) $\left(\frac{d^2y}{dx^2}\right) + 5x\left(\frac{dy}{dx}\right) - 6y = \log x$

Formation of differential equations

Algorithm

Step1. Write the given equation involving independent variable x, dependent variable y and the arbitrary constants.

Step2. Obtain the number of arbitrary constants.

Step3. Differentiate the relation in step 1 n times w.r.t. x.

Step4. Eliminate arbitrary constants with the help of n equation involving differential coefficients obtainend in step III. and in step I.

Practice problems:

4) From the differential equation representing the family of curves $y = A\cos(x + B)$ where A and B are parameters.

5) From the differential equation of the family of curves $y = a\sin(bx + c)$, a and c being parameters.

6) From the differential equation corresponding to $y^2 = a(b - x)(b + x)$ by eliminating parameters a and b.

7) Find the differential equation of all circles touching the.

a. x-axis at the origin b. y-axis at the origin

8) Find the differential equation of all the circles in the first quadrant which touch the coordinate axes.

9) From the differential equation of family of parabolas having vertex at the origin and axis along positive y-axis.

10) From the differential equation of the family of ellipses having foci on y-axis and centre at the origin.

11) Show that the differential equation representing one parameter family of curves :

$$\left(x^2 - y^2\right) = c\left(x^2 + y^2\right)^2 is \left(x^3 - 3xy^2\right)dx = \left(y^3 - 3x^2 y\right)dy$$

12) Represent the following family of curves by forming the corresponding differential equations (a, b are parameters) :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

13) Obtain the differential of all circles of radius r.

14) Find the differe4ntial equation of

a. All non-horizontal line in a plane

b. all non-vertical lines in a plane

15) Find the differential equation of the family of curves, $x = A\cos$ nt+ B sin nt t, where A and B are arbitrary constants.

16) From the differential equation corresponding to $y^2 - 2ay + x^2 = a^2$ by eliminating a.

17) From the differential equation having $y = \left(\sin^{-1}\right)^2 + A\cos^{-1} x + B$ where A and B are arbitrary constants, as its general solution.

18) From the differential equation of the family of curve represented by the equation (a being the parameter):

a. $(x-a)^2 + 2y^2 = a$ b. $y^2 = 4a(x-b)$

19) From the differential equation representing the family of ellipses having centre at the origin and foci on x-axis.

20) Form the differential equation of the family of hyperbolas having foci on x-axis and centre at the origin.

21) From the differential equation of the family of circles in the second quadrant and touching the coordinate axes.

Solution of differential equations

The solution of differential equation is a relation between the n variable involved which satisfies the differential equation.

General solution: The solution which contains as many as arbitrary constants as the order of the differential equation is called general solution of the differential equation.

Example $y = A\cos x + B\sin x$ is the general solution of differential equation $\frac{d^2y}{dx^2} + y = 0$.

Particular solution: Solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a particular solution.

For example , $y = 3\cos x + 2\sin x$. Is a particular solution.

Practice problems:

22) Show that $xy = ae^x + be^{-x} + x^2$ is a solution of the differential equation $x\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right) - xy + x^2 - 2 = 0$

23) Verify that the function $y = C_1e^{ax}\cos bx + C_2e^{ax}\sin bx, C_1, C_2$ are

arbitrary constants is a solution of the differential equation

$$\frac{d^2y}{dx^2}-2a\frac{dy}{dx}+\left(a^2+b^2\right)=0$$

24) Show that the function $y=A\cos x+B\sin x$ is a solution of the differential equation $\frac{d^2y}{dx^2}+y=0$.

25) Show that the function $y=A\cos 2x-B\sin 2x$ is a solution of the differential equation $\frac{d^2y}{dx^2}+4y=0$.

26) Show that $y=e^x(A\cos x+B\sin x)$ is the solution of the differential equation $\left(1+x^2\right)\frac{d^2y}{dx^2}-\left(2x-1\right)\frac{dy}{dx}=02y=0$.

27) Verify that $y=ce^{\tan^{-1}x}$ is a solution of the differential equation $\left(1+x^2\right)\frac{d^2y}{dx^2}-\left(2x-1\right)\frac{dy}{dx}=0$.

28) Verify that $y=e^{m\cos^{-1}x}$ is a solution of the differential equation $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=0, y(0)=1, y'(0)=2$.

29) Verify that $y=\log(x+\sqrt{x^2+a^2})^2$ is a solution of the differential equation.

Initial value problem:

30) Show that $y=x^2+2x+1$ is the solution of the initial value problems $\frac{d^3y}{dx^3}=0, y(0)=1, y'(0)=2, y''(0)=2$

31) $\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}-2y=0, y(0)=2, y'(0)=3y=e^x+e^{-x}$.

32) $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=0, y(0)=1, y'(0)=2y=xe^x$.

33) $\dfrac{dy}{dx}+y=2, y(0)=0$.

34) y=1, y(1)=0.

Differential equation of first order and first degree

The general form of a first order and first degree differential equation is $f(x, y, dy/dx)=0$

In this section we shall discuss several type of obtaning solution of differential equations.

(i) Differential equation of the form $dy/dx=f(x)$

(ii) Differential equation of the form $dy/dx=f(x)$

(iii) Differential equation in variable separable form.

(iv) Homogeneous differential equation

(v) Linear differential equation.

Type1. Differential equation of the type $\dfrac{dy}{dx}=f(x)$

We have, $dy/dx=f(x) \Leftrightarrow dy=f(x)dx$

Integrate both sides,we obtain $\int dy=\int f(x)dx+C$ or,y=$\int f(x)dx+C$

Practice problems

Solve the following differential equations:

1. $\left(e^x+e^{-x}\right)\dfrac{dy}{dx}=\left(e^x-e^{-x}\right)$

2. $\frac{dy}{dx}=\frac{1-\cos x}{1+\cos x}$

3. $\left(1+x^2\right)\frac{dy}{dx}-x=2\tan^{-1}x$

4. $\left(x^3+x^2+x+1\right)\frac{dy}{dx}=2x^2+x$

5 . $\frac{dy}{dx}+1=0; y(-1)=0$

6. $x\left(x^2-1\right)\frac{dy}{dx}=1; y(2)=0$

7. $\sqrt{a+xdy}+xdx=0$

8. $\frac{dy}{dx}=xe^x-5/2+\cos^2 x$

9. $e^{(dy/dx)}=x+1; y(0)=5$

10. $\sin(dy/dx)=k; y(0)=1$

TypeII. differential equation of the type $\frac{dy}{dx}=f(y)$

We have, dy/dx=f(y)

$\Rightarrow dx/dy=1/f(y), f(y)\neq 0$

$\Rightarrow dx=1/f(y)dy$

Integrate both sides,we obtain $\int f(x)dx=\int g(y)dy+C$

Solve the following differential equations:

1. $\frac{dy}{dx}=\frac{1}{y^2+\sin y}$

2. $\frac{dy}{dx} = \sec y$

3. $\frac{dy}{dx} + 2y^2 = 0, y(1) = 1$

TypeIII. Equation in variable sepration form

$$\int f(x)dx = \int g(y)dy + C$$

1. $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$

2. $e^x \sqrt{1-y^2}dx + \frac{y}{x}dy = 0$

3. Solve the differential equation $(1+e^{2x})dy + (1+y^2)e^x dx = 0$ given that when $x = 0, y = 1$.

4. Solve the differential equation: $(1+y^2)(1+\log x)dx + dy = 0$ given that what $x = 1.y = 1$.

5. Solve the differential equation: x $(1+y^2)dx - y(1+x^2)dy = 0$, given that $y = 0, x = 1$.

6. Solve the following differential equations: $\frac{dy}{dx} = 1 + x + y + xy$

7. $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

8. Solve: $(x^2 - yx^2)dy + (y^2 + x^2y^2)dx = 0$

9. $3e^x \tan y dx + (1-e^x)\sec^2 y dy = 0$

10. (a) $\frac{dy}{dx} + \sqrt{\frac{1+y^2}{1-x^2}} = 0$ (b) $\frac{dy}{dx} = \frac{1+y^2}{1-x^2}$

11. Solve : $\frac{dy}{dx} = y \sin 2x$, it being given that y (0) = 1.
12. Find the equation of the curve passing through the point (0,/ 4)whose differential equation is $\sin x \cos y dx + \cos x \sin y dy = 0$
13. Solve the following initial value problems: $(x+1)\frac{dy}{dx} = 2e^{-y} - 1, y(0) = 0$
14. Show that the general solution of the differential equation $\frac{dy}{dx} + \frac{y^2 + y + 1}{x^2 + x + 1} = 0$ is given by- $x + y + 1 = A(1 - x - y - 2x)$ where A is a parameter.
15. Find the particular solution of the differential equation: $\log\left(\frac{dy}{dx}\right) = 3x + 4y$ given that $y = 0$ *when* x=0.
16. In a bank principal increases at the rate of 5% per year. In how many years Rs. 1000 double itself.
17. Find the equation of the curve passing through the point (0, -2) given that at any point (x, y) on the curve the product of the slope of its tangent and y coordinate of the point is equal to the x-coordinate of the point.
18. At any point (x, y) of a curve the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (-4, -3). Find the equation of the curve given that it passes through (-2, 1).
19. Find the equation of the curve through the point (1, 0) if the slope of the tangent to the curve at any point (x, y) is $\frac{y-1}{x^2 + x}$.
20. Find the equation of the curve passing through origin if the slope

of the tangent to the curve at any point (x,y) is equal to the square of the difference of the abscissa and ordinate of the point.

21. Solve the following differential equations:

a. $\sqrt{1+x^2+y^2+x^2y^2}+xy\frac{dy}{dx}=0$

b. $\frac{dy}{dx}=\frac{x(2\log x+1)}{\sin y+y\cos y}$

c. $(y+xy)dx+(x-xy^2)dy=0$

d. $\frac{dy}{dx}=1-x+y-xy$

e. $\cos ec\ \mathrm{x}\ \log\ \mathrm{y}\frac{dy}{dx}+x^2+y^2=0$

f. $y(1-x^2)\frac{dy}{dx}=x(1+y^2)$

g. $ye^{x/y}dx=(xe^{x/y}+y^2)dy, y\neq 0$

h. $\frac{dy}{dx}=y\tan 2x, y(0)=1$

i. $\frac{dy}{dx}=1+x^2+y^2+x^2y^2, y(0)=1$

j. $xy\frac{dy}{dx}=(x+2)(y+2), y(1)=-1$

k. $\frac{dy}{dx}=1+x+y^2+xy$ *when* y=0,x=0

l. 2(y+3)-xy$\frac{dy}{dx}=0, y(1)=-2$

22. Find the equation of a curve passing though the point (0, 0) and whose differential equation is $\frac{dy}{dx} = e^x \sin x$

23. For the differential equation $xy\frac{dy}{dx} = x(x+2)(y+2)$. Find the solution curve passing through the point(1,1).

24. The volume of a spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of the balloon after t seconds.

25. In a bank principal increases at the rate of r% per year. Find the value of r if Rs. 100 double itself in 10 years $(\log_e 2 = 0.6931)$.

26. In a bank principal increase at the rate of 5% per year. An amount of Rs. 1000 is deposited with this bank, how much will it worth after 10 years $\left(e^{0.5} = 1.648\right)$.

27. In a culture the bacteria count is 100000. The number is increased by 10% in 2 hours. In how many hours will the count reach 200000, if the rate of growth of bacteria is proportional to the number present.

28. If y(x) is a solution of the different equation $\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx} = -\cos x$ and y(0) = 1, then find the value of y(π/2)

Equation reducible to variable sepration form

$\frac{dy}{dx} = f(ax+by+c)\, put$ ax+by+c=v

29. (x-y)(dx+dy) = dx - dy, y(0) = -1

30. cos(x+y)dy=dx,y(0)=0.

31. $(x+y+1)^2 dy = dx,\ y(-1) = 0$

32. $\frac{dy}{dx} + 1 = e^{x+y}$

TypeIV. Homogeneous differential equations

Homogeneous function : A function $f(x, y)$ is called a homogeneous function of degree n, if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$

$$\Rightarrow f(x, y) = x^n f\left(\frac{y}{x}\right) \text{ or } f(x,y) = y^n f\left(\frac{y}{x}\right)$$

Homogeneous Differential Equation: If a first order first degree differential equation is expressible in the form $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ where f(x,y) and g(x,y) are homogeneous function of the same degree, then it is called a homogeneous differential equation.

Algorithm

Step1. Put the differential equation in the form $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$

Step2. Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Step3. Shift v on RHS and separate the variables v and x.

Step4. Integrate both sides to obtain the solution in terms of v and x. then replace v by y/x in the solution.

33. Solve the differential equation $x^2 dy + y(x + y)dx = 0$, given that y = 1, what x = 1.

34. Solve the differential equation $(x^2 - y^2)dx + 2xydy = 0$ given that y = 1 when x = 1.

35. Solve $x^2 ydx - \left(x^3 + y^3\right)dy = 0$.

36. Find the particular solution of the differential equation $(x^2 + xy)dy = (x^2 + y^2)dx$ given that y = 0 when x = 1.

37. Find the particulars solution of the differential equation $(3xy + y^2)dx + (x^2 + xy)dy = 0$; for x = 1, y = 1.

38. Solve : $xdy - ydx = \sqrt{x^2 + y^2}\,dx$.

39. Solve:

$$y\left\{x\cos\left(\frac{y}{x}\right) + y\sin\frac{y}{x}\right\}dx - x\left\{y\sin\left(\frac{y}{x}\right) - \cos\left(\frac{y}{x}\right)\right\}dy = 0$$

40. Solve : $x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$.

41. Solve: $2ye^{x/y}dx + \left(y - 2xe^{x/y}\right)dy = 0$

42. Solve the following initial value problems:

$$x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) + x - y\left(\frac{y}{x}\right) = 0,\ y(1) = \frac{\pi}{2}$$

43. Solve each of the following initial valve problems:

a. $2x^2\frac{dy}{dx} - 2xy + y^2 = 0,\ y(e) = e$

b. $2xy + y^2 - 2x^2\frac{dy}{dx} = 0,\ y(1) = 2$

44. Solve each of the following initial value problems:

$(xe^{y/x} + y)dx = xdy,\ y(1) = 1$

45. Solve the following differential equations:

a. $x^2dy + y(x+y)dx = 0$

b. $\frac{dy}{dx} = \frac{y-x}{y+x}$

c. $\frac{dy}{dx} = \frac{x+y}{x-y}$

d. $x^2\frac{dy}{dx} = x^2 - 2y^2 + xy$

e. $\frac{dy}{dx} = \frac{y}{x} - \sqrt{\frac{y^2}{x^2} - 1}$

f. $\frac{dy}{dx} = \frac{y}{x} + \sin\left(\frac{y}{x}\right)$

g. $xy\log\left(\frac{y}{x}\right)dx\left\{y^2 - x^2\log\left(\frac{x}{y}\right)\right\}dy = 0$

h. $\left(1+e^{x/y}\right)dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

i. $x\cos\left(\frac{y}{x}\right).(ydx + xdy) = y\sin\left(\frac{y}{x}\right).(xdy - ydx)$

j. $(x-y)\frac{dy}{dx} = x + 2y$

k. $\frac{dy}{dx} - \frac{y}{x} + \cos es\frac{y}{x} = 0, y(1) = 0$

l. $\left\{x\sin^2\left(\frac{y}{x}\right) - y\right\}dx + xdy = 0, y(1) = \frac{\pi}{4}$

m. $x\frac{dy}{dx}-y+\sin\left(\frac{y}{x}\right)=0, y(2)=x$

46. Find the particular solution of the differential equation $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx}=y\cos\left(\frac{y}{x}\right)+x$ given that when x=1,y=$\frac{\pi}{4}$.

47. Fdind the particular solution of the differential equation $(x-y)\frac{dy}{dx}=x+2y$, giveen that when x =1, y = 0.

48. Find the particular solution of the differential equation $\frac{dy}{dx}=\frac{xy}{x^2+y^2}$ given that y =1 when x = 0.

Linear differential equations

A differential equation is linear if the dependent variable (y) and its derivative appear in first degree the general form of a linear differential equation of first order is $\frac{dy}{dx}+Py=Q$.

Where P and Q are function of x (or constants)

For example, (i) $\frac{dy}{dx}+xy=x^3$ (ii) $x\frac{dy}{dx}+2y=x$ (iii) $\frac{dy}{dx}+2y=\sin x$ etc.

Algorithm

Step1. Write the differential equation in the form $\frac{dy}{dx}+Py=Q$ and obtain P and Q.

Step2. Find integrating factor (I.F.) given by (i.f.) I.F=$e^{\int Pdx}$

Step3. Integrate both sides of the equation w.r.t. x

$y(I.F.) = \int Q.(I.F.)dx + C$; which gives the required solution.

Practice problems:

49. Solve the differential equation $\frac{dy}{dx} - \frac{y}{x} = 2x^2$

50. Solve the differential equation $x\log x\frac{dy}{dx} + y = \frac{2}{x}\log x$

51. Solve $\frac{dy}{dx} + y\sec x = \tan x$.

52. We are given that $\cos^2 x\frac{dy}{dx} + y = \tan x$.

53. Solve $(x^2+1)\frac{dy}{dx} + 2xy = \sqrt{x^2+4.}$.

54. Solve:

a. $x\frac{dy}{dx} + y - x + xy \cot x = 0$

b. $\left(1+x^2\right)dy + 2xydx = \cot xdx$

c. $y + \frac{d}{dx}(xy) = x(\sin x + \log x)$

55. If y(t) is a solution of $y(t)......(1+t)\frac{dy}{dx} - ty = 1$, then show that y(0) = 1 then show that $y(1)=-\frac{1}{2}$.

56. Solve: $ydx - (x + 2y^2)dy = 0$

57. Solve $ydx - (x + y^3)dy = 0$

58. Solve : $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$

59. Solve each of the following initial value problems:

$(1 + y^2)dx = (\tan^{-1} y - x)dy$, y(0) = 0

60. Solve the differential equations:

a. $\frac{dy}{dx} + 2y = 6e^x$

b. $4\frac{dy}{dx} + 8y = 5e^{-3x}$

c. $\frac{dy}{dx} + \frac{4x}{x^2 + 1}y + \frac{1}{(x^2 + 1)^2} = 0$

d. $\frac{dy}{dx} + y = \sin x$

e. $\frac{dy}{dx} + 2y = \sin x$

f. $(1 + x^2)\frac{dy}{dx} + y = \tan^{-1} x$

g. $\frac{dy}{dx} + y\cot x = x^2 \cot x + 2x$

h. $(1 + x^2)\frac{dy}{dx} + y = e^{\tan -1 x}$

i. $(1+x^2)\frac{dy}{dx}-2xy=(x^2+2)\left(x^2+1\right)$

j. $\frac{dy}{dx}-y=xe$

k. $\frac{dy}{dx}+2y=xe^{4x}$

l. Solve the differential equation $(x+2y^2)\frac{dy}{dx}=y$ given that when x = 2, y = 1.

61. Find one-parameter families of solution curves of the following differential equations: (or Solve the following differential equations (a) $x\log x\frac{dy}{dx}+y=2\log x$ (b) $e^{-y}\sec^2 ydy=dx+xdy$

62. Solve each of the following initial value problems:

a. $\frac{dy}{dx}+y\cot x=4x\cos esx, y\left(\frac{\pi}{2}\right)=0$

b. $\frac{dy}{dx}+2y\cot x=\sin x; y=0$ when x=$\frac{\pi}{3}$

c. $\frac{dy}{dx}-3y\cot x=\sin 2x; y=2$ when x=$\frac{\pi}{3}$

d. $\frac{dy}{dx}+y\cot x(2-y\cos ec\ x)dx,$

e. $dy=\cos x(2-y\cos esx)dx$

63. Find the general solution of the differential equation $\frac{dy}{dx}+2y=x^2$

64. Find the general solution of the differential equation

$$\frac{dy}{dx} - y = \cot x \ .$$

65. Solve the differential equation $(y + 3x^2)\frac{dy}{dx} = x$.

66. Find the particular solution of the differential equation

$\frac{dy}{dx} + x\cot y = 2y + y^2 \cot y$, $y \neq 0$ given that x=0 when y=$\frac{\pi}{2}$.

Application of differential equations

67. Find the equation of a curve passing through the point (1, 1). If the tangent drawn at any point P (x, y) on the curve meets the coordinate axes at A and B such that P is the mid-point of AB.

68. Find the equation of the curve passing through (2, 1), if the slope of the tangent to the curve at any point (x,y) is $\frac{x^2 + y^2}{2xy}$.

69. Find the equation of the curve passing through the point (1, 1), if the perpendicular distance of the normal at P (x, y) to the curve from the origin is equal to the distance of P from the x-axis.

70. Find the equation of a curve passing through $(1, \pi / 4)$, if the slope of the tangent to the curve at any point $P(x, y) is \left(\frac{y}{x}\right) - \cos^2\left(\frac{y}{x}\right)$.

71. The population of a city increases at a rate proportional to the number of inhabitants present at any time t. if the population of the city was 200000 in 1990 and 250000 in 2000, what will be the population in 2010?

72. Find the equation of the curve passing through the point (0, 1) if the slope of the tangent to the curve at each of its point is equal to the sum of the abscissa and the product of the abscissa and the ordinate of the point.

Practice problems:

73. What is the degree of the following differential equation

$$5x\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 6y = \log x$$

74. Write the degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^2 + y\left(\frac{dy}{dx}\right)^4 + x^3 = 0$$

75. Write the differential equation representing family of curves y = mx, where m is arbitrary constant.

76. Write the degree of the differential equation $\left(1+\frac{dy}{dx}\right)^3 = \left(\frac{d^2y}{dx^2}\right)^2$

77. Write degree of the differential equation

$$\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = x^2 \log\left(\frac{d^2y}{dx^2}\right)^2$$

78. Find the sum of the order and degree of the differential equation

$$y = x\left(\frac{dy}{dx}\right)^3 + \frac{d^2y}{dx^2}$$

79. Find the solution of the differential equation: $x\sqrt{1+y^2}\,dx + y\sqrt{1+x^2}\,dy = 0$

80. Determine the order and degree (if defined) of the following differential equations:

a. $\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$

b. $\left(y'''\right)^2 + \left(y''\right)^3 + \left(y'\right)^4 + y = 0$

c. $y'' + \left(y'\right)^2 + 2y = 0$

d. $y''' + 2y'' + y' = 0$

e. $y''' + 2y'' + \sin y = 0$

f. $y''' + y^2 + e^{y'} = 0$

81. For each of the following differential equations, find a general and particular solution satisfying the given condition:

a. $\frac{dy}{dx} = \sqrt{4 - y^2}, -2 < y < 2$

b. $x(x^2 - 1)\frac{dy}{dx} = 1, y = 0$ when x=2

c. $\cos\left(\frac{dy}{dx}\right) = a, y = 1$ when x=0

d. $\frac{dy}{dx} = y$ tan,x, y=1 when x=0

82. Solve the each of the following differential equations:

a. $(x - y)\frac{dy}{dx} = x + 2y$

b. $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$

c. $ydx + x\log\left(\frac{y}{x}\right)dy - 2xdy = 0$

d. $\frac{dy}{dx} - y = \cos x$

e. $x\frac{dy}{dx} + 2y = x^2 x \neq 0$

f. $\frac{dy}{dx} + 2y = \sin x$

g. $\frac{dy}{dx} + 3y = e^{-2x}$

h. $\frac{dy}{dx} + \frac{y}{x} = x^2$

i. $\frac{dy}{dx} + (\sec x)y = \tan x$

j. $x\frac{dy}{dx} + 2y = x^2 \log x$

k. $x\log x\frac{dy}{dx} + y = \frac{2}{x}\log x$

l. $(1+x^2)dy = 2xydx = \cot xdx$

m. $\left(x+y\right)\frac{dy}{dx} = 1$

n. $ydx + (x - y^2)dy = 0$

o. $(x + 3y^2)\frac{dy}{dx} = y$

83. Find a particulars solution of each of the following differential equations:

a. $(1+x^2)\frac{dy}{dx}+2xy=\frac{1}{1+x^2}; y=0, when$

b. $(x+y)dy+(x-y)dx=0; y=1$ when x=1

c. $x^2dy+(xy+y^2)dx=0; y=1$ when x=1

84. Find the equation of the curve passing through the point (1, 1) whose differential equation is $xdy=(2x^2+1)dx, x\neq 0$

85. Find the equation of a curve passing through the point (-2, 3), given that the slope of the tangent to the curve at any point (x, y) is $\frac{2x}{y^2}$

86. Find the equation of a curve passing through the point (0, 0) and whose differential equation is $\frac{dy}{dx}=e^x \sin x$

87. At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (-4, -3). Find the equation of the curve given that it passes through (-2, 1).

88. Show that the family of curve for which the slope of the tangent at any point (x, y) on it is is $\frac{x^2+y^2}{2xy}$ given by $x^x - y^2 = Cx$

89. Find the equation of a curve passing thought the point (0, 1). If the slope of the tangent to the curve at any point (x, y) is equal to the sum of the x-coordinate and the product of the x-coordinate and y-coordinate of that point.

90. Find the equation of the curve passing through the origin given that the slope of the tangent to the curve at any (x, y) is equal to the sum of the coordinates of the point.
91. Find the equation of the curve passing through the point (0, 2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.
92. The slope of the tangent to the curve at any point is the reciprocal of twice the ordinate at that point. The curve passes through the point (4, 3). Determine its equation.

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20 Algebra of Vectors

Introduction

Scalars and vectors

Those quantities which have only magnitude and no direction , called scalar quantities or scalars.

Those quantities which have magnitude as well as direction are called vector quantities or vectors.

Represention of vector

A directed line segment has magnitude as well as direction , so it is called vector denoted as $\overrightarrow{AB}$ or simply as $\vec{a}$. Here the point A from where the vector $\overrightarrow{AB}$ starts is called its initial point and the point B where its ends is called its terminal point.

Magnitude of a vector

The length of the vector $\overrightarrow{AB}$ or $\vec{a}$ is called magnitude of $\overrightarrow{AB}$ or $\vec{a}$ and it is represented by $|\overrightarrow{AB}|$

Position vector

Let O be the origin and P be a point in space having coordinates (x,y,z) w. r. t. the origin O. then, the vector $\overrightarrow{OP}$ or $\vec{r}$ is called the position vector of the point P w.r.t. origin O. The magnitude of $\overrightarrow{OP}$ or $\vec{r}$ is given by $\overrightarrow{OP} = \left|\vec{r}\right| = \sqrt{x^2 + y^2 + z^2}$

Components of a vector in three dimensions

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points . then,

$\overrightarrow{PQ}$ = Position vector of Q – Position vector of

$$\Rightarrow \overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Types of vectors

Null vector or zero vector

A vector, whose initial and terminal points coincide and magnitude is zero, is called a null vector and denoted as $\vec{0}$.

Unit vector

A vector of unit length is called unit vector. The unit vector in the direction of is $a = \frac{\vec{a}}{|\vec{a}|}$

Coinitial vector

Two or more vectors having the same initial point are called coinitial vectors.

Equal vector

Two vectors are said to be equal , if they have equal magnitude and same direction regardless of the position of their initial points.

Collinear vectors

Two or more vectors are said to be collinear, if they are parallel to the same line, i.e, $\vec{a}$ and $\vec{b}$ are collinear, when $\vec{a} = \lambda\vec{b}$,where λ is some scalar.

Addition of vectors

1.Triangle law of vector addition

If two vectors are represented along two sides of a triangle taken in order , then their resultant is represented is by the third side taken in opposite direction, i.e. in $\triangle ABC$, by triangle law of vector addition , we have $\overrightarrow{AB} + \overrightarrow{CA} = \overrightarrow{BA}$.

2. Parallelogram law of vector addition

If two vectors are represented along the two adjacent sides of a parallelogram, then their resultant is represente4d by the diagonal of the parallelogram through the common point. In the parallelogram OABC, we have $\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OB}$.

Properties of vector addition

(i) Commutative For vectors $\vec{a}$ and $\vec{b}$,we have $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

(ii) Associative For vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$,we have

$\vec{a}+\left(\vec{b}+\vec{c}\right)=\left(\vec{a}+\vec{b}\right)+\vec{c}$

(iii) Additive identity For vectors $\vec{a}$, a zero vector $\vec{0}$ is its additive identity as $\vec{a}+\vec{0}=\vec{a}$

(iv) Additive inverse For a vector $\vec{a}$ a negative vector of $\vec{a}$ is its additive inverse as $\vec{a}+\left(-\vec{a}\right)=\vec{0}$

Subtraction of vectors

For vectors $\vec{a}$ and $\vec{b}$,we have $\vec{a}-\vec{b}=\vec{a}+\left(-\vec{b}\right)$

Multiplicaeion of vector by a scalar

If $\vec{a}$ be a vector and m be a scalar then multiplication of vector by scalar is denoted as $\vec{a}.m$

Properties of scalar multiplication

forvectors $\vec{a}$ and $\vec{b}$, and p, q we have

(i) $p\left(\vec{a}+\vec{b}\right)=p\vec{b}+p\vec{a}$

(ii) $(p+q)\vec{a}=p\vec{a}+p\vec{a}$

(iii) $p\left(q\vec{a}\right)=(pq)\vec{a}$

Section formulae

Theorem:

(Internal Division) Let A & B be two points with position vector $\bar{a}$ and $\bar{b}$ respectively, and let C be a point dividing AB internally, in the ratio m:n then, the position vector of C is given by $\overline{OC}=\frac{m\vec{b}+n\vec{a}}{m+n}$ = .

Theorem: (External division)

Let A & B be two points with position vector $\vec{a}$ and $\vec{b}$ respectively, and let C be a point dividing AB externally, in the ratio m:n then , the position vector of C is given by $\overline{OC} = \frac{m\vec{b} - n\vec{a}}{m-n}$.

Practice problem

1. Find the position vector of a point R which divides the line segment joining P and Q whose position vector are $2\vec{a}+\vec{b}$ and $\vec{a}-3\vec{b}$ externally in the ratio 1 : 2. Also, show that P is the min-point of the line segment RQ.
2. If $\vec{a}$ and $\vec{b}$ are the position vectors of A and B respectively, find the position vector of a point C on BA produced such that BC = 1.5 BA.
3. Find the position vector of a point R which divides the joining the two points P and Q with position vectors $\overrightarrow{OP} = 2\vec{a}+\vec{b}$ and $\overrightarrow{OQ} = \vec{a}-2\vec{b}$,respectively in the ration 1 : 2 internally and externally.
4. Find a vector of magnitude 5 units which is parallel to the vector $2\hat{i}+4\hat{j}$.
5. Write down a unit vector in XY – plane, making an angle of 30° with the positive direction of x – axis.
6. A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.
7. Find the value of x, y and z so that the vectors $\vec{a} = x\hat{i}+2\hat{j}+z\hat{k}$ and $\vec{b} = 2\hat{i}+y\hat{j}+\hat{k}$ are equal.
8. Find the magnitude of the vector $\vec{a} = 3\hat{i}-2\hat{j}+6\hat{k}$

9. Show that the points A, B and C with position vectors $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$ respectively, from the vertices of a right angled triangle.

10. Find the unit vector in the direction of $\vec{a} + \vec{b}$ if $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$

11. Find the unit vector in the direction of $\overrightarrow{PQ}$, where P and Q are the points (1, 2, 3) and (4, 5, 6) respectively.

12. Find a vector of magnitude 11 in the direction opposite to that of $\overrightarrow{PQ}$, where P and Q are the points (1, 2, 3) and (-1, 0, 8) respectively.

13. The adjacent sides of a parallelogram are represented by the vectors $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} + 2\hat{k}$ Find unit vectors parallel to the diagonals of the parallelogram.

14. Find the position vector of a point R which divides the line segment joining points $P(\hat{i} + 2\hat{j} + \hat{k})$ and $Q(-\hat{i} + \hat{j} + \hat{k})$ in the ratio 2 : 1. **i.** Internally **ii.** Externally

15. Show that the points $A(2\hat{i} - \hat{j} + \hat{k})$, $B(\hat{i} - 3\hat{j} - 5\hat{k})$, $C(3\hat{i} - 4\hat{j} - 4\hat{k})$ are the vertices of a right angled triangle.

16. Find the position vector of the mid-point of the vector joining the points P (2, 3, 4) and Q (4, 1, - 2).

17. Find the value of x for which $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector.

18. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$ find a unit vector parallel to $2\vec{a} - \vec{b} + \vec{c}$.

19. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$ find a vector of

magnitude 6 units which is parallel to the vector $2\vec{a}-\vec{b}+\vec{c}$.

20. Find the vector of magnitude of 5 units parallel to the resultant of the vectors $\vec{a}=2\hat{i}+3\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+k$

21. The two vectors $\hat{j}+\hat{i}$ and $3\hat{i}-\hat{j}+4\hat{k}$ represent the sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively of triangle ABC. Find the length of the median through A.

22. Show that the vectors $2\hat{i}-3\hat{j}+4\hat{k}$ and $-4\hat{i}+6\hat{j}-8\hat{k}$ are collinear.

23. The position of vectors of the points P, Q are $\hat{i}+2\hat{j}+3\hat{k}$, $-2\hat{i}+3\hat{j}+5\hat{k}$ and $7\hat{i}-\hat{k}$ respectively. Prove that P, Q and R are collinear points.

24. If the points (-1, -1, 2) , (2, m, 5) and (3, 11, 6) are collinear, find the value of m.

25. Show that the points A (1, -2, - 8), B (5, 0, -2) C (11, 3, 7) are collinear, and find the ration in which B divides AC..

26. Using vectors show that the points A (-2, 3, 5), B (7, 0, -1), C (-3, -2, -5) and D (3, 4, 7) are such that AB and CD intersect at the point P (1, 2, 3).

27. Using vectors, find the value of λ such that the points $(\lambda,-1,3)$, (1, -1, 3) and (3, 5, 3) are collinear.

Direction cosines and direction ratios

Direction cosine

If the angle $\alpha\ \beta\ \gamma$ made by the vector $\overrightarrow{op}$ with the positive direction of the coordinates axes OX, OY, OZ respectively then $\cos\alpha, \cos\beta, \cos\gamma$ are known as the direction cosine of $\overrightarrow{OP}$ and is denoted by l , m , n i.e, $l=\cos\alpha, m=\cos\beta\ n=\cos\gamma$

Direction ratios

Let l, m, n be direction cosine of a vector $\vec{r}$ and a , c, e be three numbers such that $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$ then direction ratios of a vector $\vec{r}$ are proportional to a, b, c.

Practice problems

28. Find the position vector of a point A in space such that $\overrightarrow{OA}$. It is given that $\overrightarrow{OA}$ is inclined at 60° to OX and at 45° to OY.

29. Show that the $\hat{i}+\hat{j}+\hat{k}$ is equally inclined with the axes OX, OY and OZ.

30. Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}$.

31. If a unit vector $\vec{a}$ makes an angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$, $\hat{j}$ with and an acute angle θ with $\hat{k}$, then find θ and hence, the components of $\vec{a}$.

32. Find a vector $\vec{r}$ of magnitude $3\sqrt{2}$ units which makes an angle or $\frac{\pi}{4}$ and $\frac{\pi}{2}$ with y and z – axis respectively.

33. A vector $\vec{r}$ is inclined at equal angles to the three axes. If the magnitude of $\vec{r}$ is $2\sqrt{3}$, find.

34. If $\vec{a}$, $\vec{b}$, $\vec{c}$ represent the sides of a triangle taken in order, then write the value of $\vec{a}+\vec{b}+\vec{c}$

35. Write a unit vector in the direction of $\vec{a}=3\hat{i}-2\hat{j}+6\hat{k}$

36. If $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} + 9$ find a unit vector parallel to $\vec{a} + \vec{b}$.
37. Write the unit vector in the direction of $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$.
38. Find the position vector of the mid-point of the line segment AB, where A is the point (3, 4, -2) and B is the point (1, 2, 4).
39. Find a vector in the direction of $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude of 6 units.
40. Write the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$.
41. Find a unit vector in the direction of $\vec{a} = 2\hat{i} - 3\hat{j} + 6\hat{k}$.
42. For what value of 'a' the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $a\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear?
43. Write the direction cosines of the vectors $-2\hat{i} + \hat{j} - 5\hat{k}$.
44. Find the sum of the following vectors: $\vec{a} = \hat{i} - 2\hat{j}$, $\vec{b} = 2\hat{i} - 3\hat{j}$ $\vec{c} = 2\hat{i} + 3\hat{k}$.
45. Find a unit vector in the direction of the vector $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$
46. If $\vec{a} = x\hat{i} + 2\hat{j} - z\hat{k}$ and $\vec{b} = 3\hat{i} - y\hat{j} + \hat{k}$ are two equal vectors, then write the value of x + y + z.
47. Write a unit vector in the direction of the sum of the vectors $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 7\hat{k}$
48. Find the value of 'p' for which the vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} - 2p\hat{j} + 3\hat{k}$ are parallel.
49. Find a vector of magnitude $5\sqrt{2}$, making an angle of $\frac{\pi}{4}$ with x-

axis, $\frac{\pi}{2}$ with y-axis and an acute angle θ with z-axis.

50. Write a unit vector in the direction of $\overrightarrow{PQ}$, where P and Q are the points (1, 3, 0) and (4, 5, 6) respectively.

51. Find a vector in the direction of vector $2\hat{i} - 3\hat{j} + 6\hat{k}$ which has magnitude 21 units.

52. If $|\vec{a}| = 4$ and $-3 \le \lambda \le 2$ then write the range of $|\lambda a|$.

53. In a triangle OAC, if B is the mid-point o side AC and $\overrightarrow{OA} = \vec{a}$ $\overrightarrow{OB} = \vec{b}$, then what is $\overrightarrow{OC}$

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21 | Scalar or Dot Product

The scalar or dot product

Definition: Let $\vec{a}.$ and $\vec{b}$ be two nonzero vectors inclined at an angle θ Then the scalar product of $\vec{a}.$ with $\vec{b}$ is denoted by $\vec{a}.\vec{b}$ and is defined by $|\vec{a}|.|\vec{b}|\cos\theta$

Thus, $\vec{a}.\vec{b} = |\vec{a}|.|\vec{b}|\cos\theta$

Projection of vectors

(i) projection of $\vec{b}$ on $\vec{a}$ = $\dfrac{\vec{a}.\vec{b}}{|\vec{a}|}$

(ii) projection of $\vec{a}$ on $\vec{b}$ = $\dfrac{\vec{a}.\vec{b}}{|\vec{a}|}$

Propertiec of scalar product

I. **Commutative:** The scalar product of two vectors is commutative i.e. $\vec{a}.\vec{b} = \vec{b}.\vec{a}$.

II. **Distributive:** The scalar product of two vectors is distributive over vector addition i.e,

(i) $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$

(ii) $(\vec{b}+\vec{c}).\vec{a} = \vec{b}.\vec{a} + \vec{c}.\vec{a}.$

III. Let $\vec{a}$ & $\vec{b}$ be two non zero vectors , then $\vec{a}$ is perpendicular to $\vec{b}$ $\Leftrightarrow \vec{a}.\vec{b} = 0$

IV. For any vector $\vec{a}$, we hav $\vec{a}.\vec{a} = |\vec{a}|^2$

V. If m is scalar and $\vec{a}$ & $\vec{b}$ be any two vectors , then

$(m.\vec{a}).\vec{b}$ =m$(\vec{a}.\vec{b})$=$\vec{a}.(m.\vec{b})$

Scalar product in terms of components

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ be two vectors then,

$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ $\left[\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1\right]$

Algle Between Two Vectors

Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors inclined at an angle θ The

$\vec{a}.\vec{b} = |\vec{a}|.|\vec{b}|\cos\theta$

$$\Rightarrow \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|}$$

$$\theta = \cos^{-1}\left\{\frac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|}\right\}$$

Practice Problem

1. Find the $(\vec{a} + 3\vec{b}).(2\vec{a} - \vec{b})\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} - \hat{k}$
2. If $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 5\hat{j}$, $3\hat{i} + 2\hat{j} - 3\hat{k}$ and $\hat{i} - 6\hat{j} - \hat{k}$ respectively are position vectors of points A, B, C and D, then find the angle between the straight lines AB and CD. Deduce the $\overline{AB}$ and $\overline{CD}$ are collinear.
3. Let $\vec{a}$ and $\vec{b}$ be two vectors of the same magnitude such that the angle between then is 60° and . $\vec{a}.\vec{b} = 9$. Find $|\vec{a}|$ and $|\vec{b}|$.
4. For any two vectors $\vec{a}$ and $\vec{b}$, prove that:

   $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 \Leftrightarrow \vec{a}.\vec{b}$ are orthogonal.
5. Two vectors $\vec{a}$ and $\vec{b}$, prove that the vector $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ is orthogonal to the vector $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$.

6. If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, $|\vec{a}|=3$, $|\vec{b}|=5$, $|\vec{c}|=7$ find the angle between $\vec{a}$ and $\vec{b}$.

7. (Cauchy-Schwarz inequality) For any two vectors $\vec{a}$ and $\vec{b}$, prove that $\left(\vec{a}.\vec{b}\right)^2 \leq |\vec{a}|^2|\vec{b}|^2$ and hence show that

   $\left(a_1b_1+a_2b_2+a_3b_3\right) \leq \left(a_1^2+a_2^2+a_3^2\right)\left(b_1^2+b_2^2+b_3^2\right)$

8. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}+\vec{b}|=|\vec{a}|$, then prove that $2\vec{a}+\vec{b}$ is perpendicular to.

9. For any two vectors and prove that $|\vec{a}+\vec{b}| \leq |\vec{a}|+|\vec{b}|$
   (Triangle inequality)

10. The scalar product of the vector $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of the vector $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to 1. Find the value of λ and hence find the unit vector along $\vec{a}+\vec{c}$.

11. Show that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$, $3\hat{i}-4\hat{j}-4\hat{k}$ form the sides of a right angled triangle.

12. Show that the point A, B, C with position vectors $\vec{b}=2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$,and $3\hat{i}-4\hat{j}-4\hat{k}$ respectively, are the vertices of a right angled triangle. Also, find the remaining angles of the triangle.

13. If with reference to a right handed system of mutually perpendicular unit vector $\hat{i},\hat{j},\hat{k}$, we have $\vec{\alpha}=3\hat{i}-\hat{j}$ and $\vec{\beta}=2\hat{i}+\hat{j}-3\hat{k}$. Express $\vec{\beta}=\vec{\beta_1}+\vec{\beta}_2$, where $\vec{\beta_1}$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

14. Find the values of x for which the angle between the vector $\vec{\alpha} = 2x^2\hat{i} + 4x\hat{j} + \hat{k}$ and $\vec{\alpha} = 7\hat{i} - 2\hat{j} + x\hat{k}$ is obtuse.

15. If $\vec{a}, \vec{b}, \vec{c}$ are three mutually perpendicular sectors of equal magnitude, prove that $\vec{a} + \vec{b} + \vec{c}$ is equally inclined with vector, $\vec{a}, \vec{b}, and\ \vec{c}$. Also find angle.

16. Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors of magnitudes 3, 4 and 5 respectively. If each one is perpendicular to the sum of the other two vectors, prove that $\left|\vec{a} + \vec{b} + \vec{c}\right| = 5\sqrt{2}$.

17. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, find the value of $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$.

18. Three vector $\vec{a}.\vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. Evaluate the quantity $\mu\, \vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$ $|\vec{a}|=1, |\vec{b}|=4, |\vec{c}|=2$.

19. (i) Dot product of a vector with $\hat{i} + \hat{j} - 3\hat{k}$, $\hat{i} + 3\hat{j} - 2\hat{k}$ and $2\hat{i} + \hat{j} - 4\hat{k}$ and 0,5and 8 respectively. Find the vector.

(ii) Don product of a vector with $\hat{i} - \hat{j} + \hat{k}$, $2\hat{i} + \hat{j} - 3\hat{k}$ and $\hat{i} + \hat{j} + \hat{k}$ are respectively 4, 0 and 2. Find the vector.

20. If $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + \hat{j} - 2\hat{k}$ and $\vec{c} = \hat{i} + 3\hat{j} - \hat{k}$ find λ such that $\vec{a}$ is perpendicular to $\lambda\vec{b} + \vec{c}$

21. If $\vec{p} = 5\hat{i} + \lambda\hat{j} - 3\hat{k}$ and $\vec{q} = \hat{i} + 3\hat{j} - \hat{k}$ then find the value of λ , so that $\vec{p} + \vec{q}$ and $\vec{p} - \vec{q}$ are perpendicular vectors.

22. If $\vec{\alpha} = 3\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{\beta} = 2\hat{i} + \hat{j} - 4\hat{k}$ then express $\vec{\beta}$ in the form of $\vec{\beta} = \vec{\beta_1} + \vec{\beta_2}$, where $\vec{\beta_1}$ is parallel $\vec{\alpha}$ and $\vec{\beta_2}$ is perpendicular to $\vec{\alpha}$.

23. If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}.\vec{b}=\vec{0}$. But, the converse need not be true. Justify your answer with an example.

24. Show that the vectors $\vec{a}=3\hat{i}-2\hat{j}+\hat{k}$, $\vec{b}=\hat{i}-3\hat{j}+5\hat{k}$, $\vec{c}=2\hat{i}+\hat{j}-4\hat{k}$ form a right angled triangle.

25. If $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}$,$\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

26. Find the magnitude of two vectors $\vec{a}$ *and* $\vec{b}$, having the same magnitude and such that the angle between them is $60°$ and their scalar product is ½.

27. Find the projection of $\vec{b}+\vec{c}$ on $\vec{a}=2\hat{i}-2\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$.

28. If $\vec{a}=5\hat{i}-\hat{j}-3\hat{k}$ and $\vec{b}=\hat{i}+3\hat{j}-5\hat{k}$ then show that the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are orthogonal.

29. A unit vector $\vec{a}$ makes angles $\frac{\pi}{4}$ and $\frac{\pi}{3}$ with $\hat{i}$ and $\hat{j}$ respectively and an acute angle θ with $\hat{k}$. Find the angle θ and components of $\vec{a}$.

30. If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=2$, $|\vec{b}|=1$ and $\vec{a}.\vec{b}=1$ then find the value of $(3\vec{a}-5\vec{b}).(2\vec{a}+7\vec{b})$

31. If $\vec{a}$ is a unit vector, then find $|\vec{x}|$ in each of the following:

    (i) $(\vec{x}-\vec{a}).(\vec{x}+\vec{a})=8$ (ii) $(\vec{x}-\vec{a}).(\vec{x}+\vec{a})=12$

32. Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a}+\vec{b}).(\vec{a}-\vec{b})=8$ and $|\vec{a}|=8|\vec{b}|$

33. Find the angle between two vectors $\vec{a}$ and $\vec{b}$, if $|\vec{a}|=\sqrt{3}$, $|\vec{b}|=2$ and $\vec{a}.\vec{b}=\sqrt{b}$.

34. Express the vector $\vec{a} = 5\hat{i} - 2\hat{j} + 5\hat{k}$ as the sum of two vectors such that one is parallel to the vector $\vec{b} = 3\hat{i} + \hat{k}$ and other is perpendicular to $\vec{b}$.

35. Let . $\vec{a}.\vec{a} = 0$ and $\vec{a}.\vec{b} = 0$, what can you conclude about the vector $\vec{b}$?

36. Let $\vec{u}, \vec{v},$ and $\vec{w}$ be vector such $\vec{u} + \vec{v} + \vec{w} = \vec{0}$ if $|\vec{u}|=3$ $|\vec{v}|=4$ and $|\vec{w}|=5$, then find $\vec{u}.\vec{v} + \vec{v}.\vec{w} + \vec{w}.\vec{u}$.

37. What is the angle between vectors $\vec{a}$ and $\vec{b}$ with magnitudes 2 and $\sqrt{3}$ respectively ? Given $\vec{a}.\vec{b} = \sqrt{3}$.

38. Find the angle between the vectors $\vec{a} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$.

39. For what value of λ are the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ perpendicular to each other?

40. Find the projection of $\vec{a}$. on $\vec{b}$, $\vec{a}.\vec{b} = 8$ and $\vec{b} - 2\hat{i} + 6\hat{j} + 3\hat{k}$.

41. Write the value of p for which $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} - 4\hat{k}$ are parallel vectors.

42. Find the value of λ if the vectors $2\hat{i} + \lambda\hat{j} + 3\hat{k}$ and $3\hat{i} + 2\hat{j} - 4\hat{k}$ are perpendicular to each other.

43. If $|\vec{a}| = 2$, $|\vec{b}|=3$ and $\vec{a}.\vec{b} = 3$ find the projection of on

44. Write the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2 respectively having $\vec{a}.\vec{b} = \sqrt{6}$

45. Find λ , when the projection of $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ is 4 units.

46. For what value of λ are the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ perpendicular to each other?

47. Write the projection of the vector $7\hat{i} + \hat{j} - 4\hat{k}$ on the vector $2\hat{i} + 6\hat{j} + 3\hat{k}$.

48. Write the value of λ so that the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ are perpendicular to each other.

49. Write the projection of $\vec{b} + \vec{c}$ on $\vec{a}$, when $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$.

50. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors $|\vec{a} + \vec{b}| = 3$ and $|\vec{b}| = 5$ find the value of $\vec{b}$.

51. If $\vec{a}$ and $\vec{b}$ vector are such that $|\vec{a}| = 3$, $|\vec{b}| = \frac{2}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector, then write the angle between $\vec{a}$ and $\vec{b}$.

52. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a} + \vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$.

53. If $\vec{a}$ and $\vec{b}$ are unit vectors, then find the angle between $\vec{a}$ and $\vec{b}$, given that is unit vector $\left(\sqrt{3}\vec{a} - \vec{b}\right)$.

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22 | Vector or Cross Product

Defination

Let $\vec{a}, \vec{b}$ be two non zero nonparallel vectors. then the vector cross produc $\vec{a} \times \vec{b}$ is given by such that $\vec{a}\times\vec{b} = |\vec{a}|.\,|\vec{b}|\sin\theta\, \hat{n}$ such that $0 \le \theta \le \pi$ where $\hat{n}$ is a unit vector is perpendicular to both $\vec{a} \,\&\, \vec{b}$.

Properties of cross product of two vectors

1. Unit vector $\hat{n} = \dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.
2. For vectors $\vec{a}, and\ \vec{b}$, if $\vec{a} \times \vec{b} = \vec{0}$, $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \,||\, \vec{b}$.
3. Angle between two non-zero vectors is given by

   $$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|.|\vec{b}|} \text{ or } \Rightarrow \theta = \sin^{-1}\left\{\frac{|\vec{a}\times\vec{b}|}{|\vec{a}|.|\vec{b}|}\right\}.$$

4. $\vec{a}\times\vec{a} = \vec{0}$.
5. $\vec{a} \times \vec{b} = \overline{\vec{b}\times\vec{a}}$.
6. $\vec{a} \times \left(\vec{b}+\vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.
7. $\lambda\left(\vec{a} \times \vec{b}\right) = \left(\lambda\vec{a}\right)\times\vec{b} = \vec{a} \times \left(\lambda\vec{b}\right)$.
8. If $\vec{a}$ is parallel to $\vec{b}$, then $\vec{a} \times \vec{b} = \vec{0}$ and converse is true.

   Or

   Two non zero vectors $\vec{a}$ and $\vec{b}$ are collinear iff $\vec{a} \times \vec{b} = \vec{0}$
9. If $\theta = \frac{\pi}{2}$, then $= |\vec{a} \times \vec{b}| = |\vec{a}|.|\vec{b}|$.

10. If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram , then its area is given by $\left|\vec{a} \times \vec{b}\right|$.

11. If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle , then its area is given by $\frac{1}{2}\left|\vec{a} \times \vec{b}\right|$.

12. If $\vec{d_1}$ and $\vec{d_2}$ represent the diagonal of a parallelogram , then its area is given by $\frac{1}{2}|\vec{d_1} \times \vec{d_2}|$.

13. $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$

    And $\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j}$.

14. $\hat{j}\times\hat{i} = -\hat{k},\ \hat{k}\times\hat{j} = -\hat{i},\ \hat{i}\times\hat{k} = -\hat{j}$.

15. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{a} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ Then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} + (b_1a_3 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

16. If A , B , C are the position vector of plane ABC, is $\frac{\vec{AB}\times\vec{AC}}{|\vec{AB}\times\vec{AC}|}$

Practice Problems

1. Find λ and μ if $\left(2\hat{i} + 6\hat{j} + 27\hat{k}\right)\times\left(\hat{i} + \lambda\hat{j} + \mu\hat{k}\right) = \vec{0}$.

2. If $\vec{a} = l_1\hat{i} + m_1\hat{j} + n_1\hat{k}$ find the value of $\left(\vec{r}\times\hat{i}\right).\left(\vec{r}\times\hat{j}\right) + xy$.

3. Find a unit vector perpendicular to each of the vector $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ where $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

4. Let $\vec{a}=\hat{i}-\hat{j}, \vec{b}=3\hat{i}-\hat{k}$ and $\vec{c}=7\hat{i}-\hat{k}$ Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}.\vec{d}=1$.

5. Show that the area of a parallelogram having diagonals $3\hat{i}+\hat{j}-2\hat{k}$ and $\hat{i}-3\hat{j}+4\hat{k}$ is $5\sqrt{3}$ square units.

6. Find the area of the triangle whose vertices are A(3, -1, 2), B(1, -1, -3) and C (4, -3, 1). $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$ is show that $\vec{a}-\vec{d}$ is parallel to $\vec{b}-\vec{c}$, where $\vec{a}\neq\vec{d}$ and $\vec{b}\neq\vec{c}$.

7. If $\vec{a},\vec{b},\vec{c}$ are three vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then prove that. $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$.

8. Prove that the normal to the place containing three points whose position vectors are $\vec{a},\vec{b},\vec{c}$ lies in the direction $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$.

9. Prove that $\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)=1\left(\vec{a}\times\vec{b}\right)$ and interpret it geometrically.

10. For any two vectors and , show that :

$$\left(1+|\vec{a}|^2\right)\left(1+|\vec{b}|^2\right)=\left\{\left\{\left(1-\vec{a}.\vec{b}\right)^2\right\}+|\vec{a}+\vec{b}+\left(\vec{a}\times\vec{b}\right)^2|\right\}.$$

11. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \quad \vec{c}=\hat{j}-\hat{k}$ are given vectors, then find a vector satisfying the equations $\vec{a}\times\vec{b}=\vec{c}\ \vec{a}.\vec{b}=3$.

12. Find the area of the parallelogram determined by the vectors : $\hat{i}-3\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$.

13. What inference can you draw if $\vec{a}\times\vec{b}=\vec{0}$ and $\vec{a}.\vec{b}=\vec{0}$.

14. If $|\vec{a}|=\sqrt{25}$, $|\vec{b}|=7$ and $|\vec{a}\times\vec{b}|=35$ find $\vec{a}.\vec{b}$.

15. Find $\vec{a}=\hat{i}+4\hat{j}+\hat{k}$, $\vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$ Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}.\vec{d}=15$.

16. Find a unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ where $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$.

17. Using vectors find the area of the triangle with vertices, A (2, 3, 5), B (3, 5, 8) and C(2, 7, 8).

18. If $\vec{a}=2\hat{i}-3\hat{j}+\hat{k}$, $\vec{b}=-\hat{i}+\hat{k}$, $\vec{c}=2\hat{i}-\hat{k}$ are three vectors, find the area of the parallelogram having diagonalsand $\left(\vec{a}+\vec{b}\right) and \left(\vec{b}-\vec{c}\right)$

19. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}+5\hat{k}$ and $\hat{i}-4\hat{j}-3\hat{k}$ Find the unit example.

20. If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$,then $\vec{a}\times\vec{b}=\vec{0}$. Is the converse true? Justify your answer with an example.

21. If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$,$\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ then verify that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$.

22. Using vectors, find the area of the triangle with vertices :
    (i) A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5)
    (ii) A(1, 2, 3), B(2, -1, 4) and C (4, 5, -1).

23. Find all vectors of magnitude $10\sqrt{3}$ that are perpendicular to the plane of $\hat{i}+2\hat{j}+\hat{k}$ and $-\hat{i}+3\hat{j}+4\hat{k}$.

24. Write the value of $\hat{i}+\left(\hat{j}+\hat{k}\right)+\hat{j}\times\left(\hat{k}+\hat{i}\right)+\hat{k}\times\left(\hat{i}+\hat{j}\right)$.

25. Find out angle between tow vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively and when $|\vec{a}\times\vec{b}|=\sqrt{3}$.

26. Vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=\sqrt{3}$, $|\vec{b}|=\frac{2}{3}$ is a unit vector. Write the angle between . $\vec{a}$ and $\vec{b}$.

27. Find λ, if $\left(2\hat{i}+6\hat{j}+14\hat{k}\right)\times\left(\hat{i}-\lambda\hat{j}+7\hat{k}\right)=\vec{0}$.
28. Write the value of the area of the parallelogram determined by the vectors $2\hat{i}$ and $3\hat{j}$.
29. Write the value of $\left(\hat{i}\times\hat{j}\right).\hat{k}\left(\hat{j}+\hat{k}\right).\hat{j}$.
30. Find a vector of magnitude $\sqrt{171}$ which is perpendicular to both of the vectors $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}+2\hat{k}$.

Scalar Triple Product of Vectors

Definition

Suppose $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vectors. then, scalar product of $\vec{a}$ and $\vec{b}\times\vec{c}$ i.e, $\vec{a}.\left(\vec{b}\times\vec{c}\right)$ is called the scalar triple product of $\vec{a}$. $\vec{b}$ and $\vec{c}$ and it is denoted by $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

Properties Of Scalar Triple Product

For any three vectors $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$.

1. $\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.
2. If cyclic order of three vectors is unchanged then scalar triple product remains unchanged, $\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\left[\vec{b}\ \vec{c}\ \vec{a}\right]=\left[\vec{c}\ \vec{a}\ \vec{b}\right]$.
3. If cyclic order of three vectors is changed then scalar triple product changes in sign but not in magnitude, i.e

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right]=-\left[\vec{b}\ \vec{c}\ \vec{a}\right]$$

$$=-\left[\vec{c}\ \vec{a}\ \vec{b}\right]=-\left[\vec{a}\ \vec{c}\ \vec{b}\right].$$

4. $\left[\vec{a}\ \vec{a}\ \vec{b}\right]=\left[\vec{b}\ \vec{b}\ \vec{a}\right]=\left[\vec{c}\ \vec{c}\ \vec{b}\right]=0$.

5. $\left[k\ \ \vec{a}\ \vec{b}\ \vec{c}\right]=k\left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

6. $|\vec{a}\ .\left(\vec{b}\times\vec{c}\right)|$ gives the volume of a parallelepiped formed by adjacent sides given by the vector $\vec{a}$, $\vec{b}$ & $\vec{c}$.

7. $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vectors are coplanar, if and only if $\vec{a}.\left(\vec{b}\ \times\ \vec{c}\right)=0$.

8. A, B, C, & D are coplanar , if $\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right]=0$.

Practice Problem

31. For any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ prove that $\left[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\ \right]=2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

32. Show that vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar $\vec{a}+\vec{b}$, $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$ are coplanar.

33. Show that four points whose position vectors are $6\hat{i}-7\hat{j}$,$16\hat{i}-19\hat{j}-4\hat{k}$, $3\hat{i}-6\hat{k}$, $2\hat{i}-5\hat{j}+10\hat{k}$.

34. Find $\vec{a}\ .\left(\vec{b}\times\vec{c}\right)$ if $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}+2\hat{k}$.

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23 Directions Cosines And Direction Ratios

Direction osines and direction ratio

Definition

If a directed line OP makes angle α, β, γ with positive X axis, Y axis and Z axis respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called direction cosines of a line . they are denoted by l, m, n. therefore,

$l = \cos\alpha \quad m = \cos\beta \quad n = \cos\gamma$. Also sum of square of direction cosine of a line is always 1 i.e.,

$$l^2 + m^2 + n^2 = 1 \text{ or } \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Direction cosines of X, Y, Z axis are (1,0,0) , (0,1,0), (0,0,1).

Direction Ratios Of A Line

Any three numbers proportional to the direction cosines of a line are called direction ratios of a line,

1. $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$
2. If a, b, care direction ratios of a line , then its direction cosines are $l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$, $m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$, $n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$
3. Direction ratios of a line PQ passing through the point $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ are x_2-x_1 , y_2-y_1, z_2-z_1 and direction cosines are $\frac{x_2 - x_1}{|PQ|}$, $\frac{y_2 - y_1}{|PQ|}$, $\frac{z_2 - z_1}{|PQ|}$
4. Angle between two vectors in terms of their direction cosines If $\vec{a} = l_1\hat{i} + m_1\hat{j} + n_1\hat{k}$ and $\vec{b} = l_2\hat{i} + m_2\hat{j} + n_2\hat{k}$ then $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

5. Angle between two vectors in terms of their direction ratios

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_1^2 + b_1^2 + c_1^2}}$$

PRACTICE PROBLEMS

1. If l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both of them are $(m_1n_2 - m_2n_1)(n_1l_2 - n_2l_1)$ $(l_1m_2 - l_2m_1)$
2. If a line makes angles of 90°, 60° and 30° with the positive direction of x, y and z- axis respectively, find its direction cosines.
3. If a line has direction ratios 2, -1, -2 determine its direction cosines.
4. Find the direction cosines of the line passing through two points (-2, 4, -5) and (1, 2, 3).
5. Using direction ratios show that the points A (2, 3, -4), B (1, -2, 3) and C(3, 8, -11) are collinear.
6. Find out direction cosines of the sides of the triangle whose vertices are (3, 5, -4), (-1, 1, 2) and (-5, -5, 2).
7. Find the distance of the point (2, 3, 4) from the x – axis.
8. If a line has direction ratios proportional to 2, -1, -2, then what are its direction consines?
9. Write direction consines of a line parallel to z – axis.
10. If a unit vector $\vec{a}$ makes an angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with and an acute angle θ with $\hat{k}$, then find the value of θ .
11. Write the distance of a point P (a, b, c) form x – axis.

Straight Line in Space

Equation of line through a given point and parrallel to a given vector

1. **Vector form =** $\vec{r} = \vec{a} + \lambda\vec{b}$ where $\vec{a}$ = position vector of a point through which the line is passing , $\vec{b}$ **=** a vector parallel to a given line.
2. **Cartesian form**The cartesian equation of a line passing through a point A(x_1, y_1,z_1) and having direction ratios a, b,. c is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

if l, m, n are the direction cosines of a line then the equation of the line is $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$

Equation Of Line Passing Through Two Given Points

1. **Vector form** $\vec{r} = \vec{a} + \lambda\left(\vec{b} - \vec{a}\right) \quad \lambda \in R$
2. **Cartesian form**The cartesian equation of a line passing through two points A(x_1, y_1,z_1) and B(x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Angle Betyween Two Lines

1. **Vector form** = $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \lambda\vec{b_2}$ is given as

$$\cos\theta = \left|\frac{\vec{b_1}.\vec{b_2}}{|\vec{b_1}|\,|\vec{b_2}|}\right|$$ where θ is acute angle between the line.

2. **Cartesian form**Angle between the lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$

and $\frac{x - x_1}{a_2} = \frac{y - y_1}{b_2} = \frac{z - z_1}{c_2}$ is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} + \sqrt{a_2^2 + b_2^2 + c_2^2}} \text{ or}$$

$$\sin\theta = \frac{\sqrt{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2} + \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Condition For Two Lines Are Perpendicular And Parallel

1. **Vector form**Two lines are perpendicular to each other if $\vec{b_1}.\vec{b_2} = 0$.
2. **Cartesian form** If $a_1a_2 + b_1b_2 + c_1c_2 = 0$ (direction ratio form)

   Or $l_1l_2 + m_1m_2 + n_1n_2 = 0$ (direction cosine form)

   1. Vector form Two lines are parallel , if $\vec{b_1} = \lambda\vec{b_2}$

   2. Cartesian form If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ (direction ratio form)

   Or $\frac{t_1}{t_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

Skew lines Lines which are neither parallel nor intersecting lines , are called skew lines . in fact , such lines are non coplanar.

Shortest Distance Between Two Lines

For skew lines the line of the shortest distance will be perpendicular to both the lines.

1. **Vector form** If the lines are $\vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \lambda\vec{b_2}$ Then ,

   shortest distance d= $\left| \frac{(\vec{b_1} \times \vec{b_2})(\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right|$

2. **Cartesian form** If the lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_1}{a_2}=\frac{y-y_1}{b_2}=\frac{z-z_1}{c_2}$ then the shortest distance

$$d=\left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{\left(a_1b_2-a_2b_1\right)^2+\left(b_1c_2-b_2c_1\right)^2+\left(c_1a_2-c_2a_1\right)^2}}\right|$$

3. Condition for two given lines to be intersect , is

$$\left(\vec{b_1}\times\vec{b_2}\right)\left(\vec{a_2}-\vec{a_1}\right)=0 \quad \text{Or} \quad \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

Distance Between Two Parallel Lines

Let the lines be $\vec{r}=\vec{a_1}+\lambda\vec{b}$ and $\vec{r}=\vec{a_2}+\mu\vec{b}$, then the distance between parallel lines is $\left|\frac{\vec{b}\times\left(\vec{a_2}-\vec{a_1}\right)}{|\vec{b}|}\right|$

1) Find the vector equation of a line passing through a point with position vector $2\hat{i}-\hat{j}+\hat{k}$ and parallel to the line joining the points $-\hat{i}+4\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+2\hat{k}$ Also, find the Cartesian equation of this equation.

2) Find the point on the line $\frac{x+2}{3}=\frac{y+1}{2}=\frac{z-3}{2}$ at a distance of $3\sqrt{2}$ from the point (1, 2, 3).

3) Find the vector and Cartesian equations of the line through the point (5, 2, -4) and which is parallel to the vector $3\hat{i}+2\hat{j}-8\hat{k}$.

4) Find the vector equation of the line passing through the points (-1, 0, 2) and (3, 4, 6) .

5) The cartesian equation of a line are $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$. Find a vector equation for the line.

6) Find a points on the line $\frac{x+2}{3}=\frac{y+1}{2}=\frac{z-3}{2}$ at a distance of 5 units from the point P(1, 2, 3).

7) Find the cartesian and vector equations of a line which passes through the point (1, 2, 3) and is parallel to the line $\frac{-x-2}{1}=\frac{y+2}{7}=\frac{2z-6}{3}$.

8) The cartesian equations of a line are $3x+1=6y-2=1-z$. Find the fixed point through which it passes, its direction ratios and also its vector equation.

9) Find the Cartesian equations of the line passing through the point (-1, 3, -2) and perpendicular to the lines $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and $\frac{x+2}{-3}=\frac{y-1}{2}=\frac{z+6}{5}$

10) A line passes through (2, -1, 3) and is perpendicular to the line $\vec{r}=\left(\hat{i}+\hat{j}-\hat{k}\right)+\lambda\left(2\hat{i}-2\hat{j}+\hat{k}\right)$ and $\vec{r}=\left(2\hat{i}-\hat{j}-3\hat{k}\right)+\mu\left(\hat{i}+2\hat{j}+2\hat{k}\right)$ Obtain its equation.

11) Find the value of λ so that the lines : $l_1:\frac{1-x}{3}=\frac{7y-14}{2\lambda}=\frac{z-3}{3}$

$l_2 : \frac{7-7x}{3\lambda} = \frac{y-5}{1} = \frac{6-z}{5}$ Also, find the equations of a line passing through the point (3, 2, -4) and parallel to line .

12) Show that the three lines with direction cosines $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}, \frac{4}{13}, \frac{12}{13}, \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular.

13) Show that the line through the points (1, -1, 2) and (3, 4, -2) is perpendicular to the through the points (0, 3, 2) and (3, 5, 6).

14) Show that the line through the points (4, 7, 8) and (2, 3, 4) is parallel to the line through the points (-1, -2, 1) and (1, 2, 5).

15) Find the cartesian equation of the line which passes through the point (-2, 4, -5) and parallel to the line given by $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

16) Show that the lines $\frac{x-5}{7} = \frac{y+7}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{6}$ are perpendicular to each other.

17) Show that the line joining the origin to the point (2, 1, 1) is perpendicular to the line determined by the points (3, 5, -1) and (4, 3, -1).

18) Find the equation of a line parallel to x-axis and passing through the origin.

19) Find the angle between and following pairs of lines: $\frac{-x+5}{-2} = \frac{y-7}{7} = \frac{z+3}{-3}$ and $\frac{x+2}{-1} = \frac{2y-8}{4} = \frac{z-5}{4}$

20) Find the angle between the pairs of lines with direction ratios proportional to a, b, c and b – c, c – a, a – b.

21) Find the equations of the passing through the points (2, 1, 3) and perpendicular to the lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and

$\frac{x}{-3}=\frac{y}{2}=\frac{z}{5}$.

22) Determine the equations of the line passing through the point (1, 2, -4) and perpendicular to the two lines $\frac{x-8}{8}=\frac{y+9}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{5}$

23) If the lines $\frac{x-1}{-3}=\frac{y-5}{2\lambda}=\frac{z-3}{2}$ and $\frac{x-1}{3\lambda}=\frac{y-1}{1}=\frac{z-6}{-5}$ are perpendicular, find the value of λ.

24) If the coordinates of the points A, B, C, D be (1, 2, 3), (4, 5, 7), (-4, 3, -6) and (2, 9, 2) respectively, then find the angle between the lines AB and CD.

25) Find the value of λ so that the following lines are perpendicular to each other $\frac{x-5}{5\lambda+2}=\frac{2-y}{5}=\frac{1-z}{-1}$, $\frac{x}{1}=\frac{2y+1}{4\lambda}=\frac{1-z}{-3}$

26) Find the direction consines of the line $\frac{x+2}{2}=\frac{2y-7}{6}=\frac{5-z}{6}$. Also, find the vector equation of the line through the point A (-1, 2, 3) and parallel to the given line.

27) Show that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$ intersect. Find their point of intersection.

28) Show that the lines $\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-1}{5}$ and $\frac{x+2}{4}=\frac{y-1}{3}=\frac{z+1}{-2}$ do not intersect.

29) Shot that the lines $\vec{r}=\left(\hat{i}+\hat{j}-\hat{k}\right)+\lambda\left(3\hat{i}-\hat{j}\right)$ and $\vec{r}=\left(4\hat{i}-\hat{k}\right)+\mu\left(2\hat{i}+3\hat{k}\right)$ intersect. Find their point of intersection.

30) Find the equations of the lines through the origin which intersect the line $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}$ at angle of $\frac{\pi}{3}$ each.

31) $\overrightarrow{AB}=3\hat{i}-\hat{j}+\hat{k}$ and $\overrightarrow{CD}=-3\hat{i}+2\hat{j}+4\hat{k}$ are two vectors. The position vectors of the points A and C are $6\hat{i}+7\hat{j}+4\hat{k}$ and $-9\hat{j}+2\hat{k}$ respectively. Find the position vector of a point P on the line AB and point Q on the line $\overrightarrow{PQ}$ is perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{CD}$ both.

32) Show that the lines $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}$ and $\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}$ intersect. Find their point of intersection.

33) Prove that the lines through A (0, - 1, -1) and B (4, 5, 1) intersects the line through C (3, 9, 4) and D (-4, 4, 4). Also find their point of intersection.

34) Prove that the line through A (0, -1, -1) and B (4, 5, 1) intersects the line through C (3,9, 4) and D (-4, 4, 4). Also, find their point of intersection.

35) $\frac{x-5}{4}=\frac{y-7}{-5}=\frac{z+5}{-5}$ and $\frac{x-8}{7}=\frac{y-4}{1}=\frac{z-5}{3}$

36) Show that the lines $\vec{r}=3\hat{i}+2\hat{j}-4\hat{k}+\lambda\left(\hat{i}+2\hat{j}+2\hat{k}\right)$ and $\vec{r}=5\hat{i}-2\hat{j}+\mu\left(3\hat{i}+2\hat{j}+6\hat{k}\right)$ are intersecting. Hence, find their point intersection.

37) Find the image of the point (1, 6, 3) in the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$. Also, write the equation of the line joining the given point and its image and find the length of the segment joining the given point and its image.

38) Find the perpendicular distance of the point (1, 0, 0) from the line $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$. Also, find the coordinates of the food of the perpendicular and the equation of perpendicular.

39) Find the foot of perpendicular from the point (2, 3, 4) to the line $\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}$ Also, find the perpendicular distance from the given point to the line.

40) Find the equation of line passing through the points A(0, 6, -9) and B (-3, -6, 3). If D is the foot of perpendicular drawn from a point C (7, 4, -1) on the line AB, then find the coordinates of the point D and the equation of line CD.

41) Find the distance of the point (2, 4, -1) from the $\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}$.

42) Find the coordinates of the foot of perpendicular drawn from the point A (1, 8, 4) to the line joining the points B (0, -1, 3) and C (2, -3, -1).

43) Find the shortest distance between the following pairs of lines whose vector equations are:

a $\vec{r}=(1-t)\hat{i}+(t-2)\hat{j}+(3-t)\hat{k}$ and $\vec{r}=(s+1)\hat{i}+(2s-1)\hat{j}-(2s+1)\hat{k}$

b $\vec{r}=\left(2\hat{i}-\hat{j}-\hat{k}\right)+\lambda\left(2\hat{i}-5\hat{j}+2\hat{k}\right)$ and $\vec{r}=\left(\hat{i}+2\hat{j}+\hat{k}\right)+\mu\left(\hat{i}-\hat{j}+\hat{k}\right)$

c $\vec{r}=\left(\hat{i}+\hat{j}\right)+\lambda\left(2\hat{i}-\hat{j}+\hat{k}\right)$ and $\vec{r}=2\hat{i}+\hat{j}-\hat{k}+\mu\left(3\hat{i}-5\hat{j}+2\hat{k}\right)$

44) Find the shortest distance between the following pairs of lines whose Cartesian equations are: $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-5}{5}$

45 $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$ and $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$

46) Write the vector equations of the following lines and hence determine the distance between them $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}$ and $\frac{x-3}{4}=\frac{y-3}{6}=\frac{z+5}{12}$

47) Find the shortest distance between the lines

a $\vec{r}=\left(\hat{i}+2\hat{j}+\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$ and $\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu\left(2\hat{i}+\hat{j}+2\hat{k}\right)$

b $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ and $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$

c $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda\left(\hat{i}-3\hat{j}+2\hat{k}\right)$ and $\vec{r}=4\hat{i}+5\hat{j}+6\hat{k}+\mu\left(2\hat{i}+3\hat{j}+\hat{k}\right)$

d $\vec{r}=6\hat{i}+2\hat{j}+2\hat{k}+\lambda\left(\hat{i}-2\hat{j}+2\hat{k}\right)$ and $\vec{r}=-4\hat{i}-\hat{k}+\mu\left(3\hat{i}-2\hat{j}-2\hat{k}\right)$

48) Find the distance between the lines given by l_1 *and* l_2

$\vec{r}=\hat{i}+2\hat{j}-4\hat{k}+\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$ and $\vec{r}=3\hat{i}+3\hat{j}-5\hat{k}+\mu\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$

49) The cartesian equations of a line AB are $\frac{2x-1}{\sqrt{3}}=\frac{y+2}{2}=\frac{z-1}{3}$. Find the direction cosines of a line parallel to AB.

50) If the equations of a line AB are $\frac{3-x}{1}=\frac{y+2}{-2}=\frac{z-5}{4}$, write the direction ratios of a line parallel to AB.

51) Write the vector equation of a line given by $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$

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52) The equations of a line are given by $\frac{-x+2}{-2}=\frac{y-1}{7}=\frac{z+3}{-3}$. Write the direction cosines of a line parallel to this line .

53) Find the Cartesian equations of the line which passes through the point (-2, 4, -5) and is parallel to the line $\frac{x+3}{3}=\frac{4-y}{5}=\frac{z+8}{6}$

54) Find the angle between the lines $\vec{r}=\left(2\hat{i}-5\hat{j}+\hat{k}\right)+\lambda\left(3\hat{i}-2\hat{j}+6\hat{k}\right)$ and $\vec{r}=7\hat{i}-6\hat{k}+\mu\left(\hat{i}+2\hat{j}+2\hat{k}\right)$

55) Find the angle between the lines 2x = 3y = -z and 6x = - y = - 4z.

56) Find the angle between following pair of lines $\frac{x+2}{-1}=\frac{2y-8}{4}=\frac{z-5}{4}$ and $\frac{x+2}{-1}=\frac{2y-8}{4}=\frac{z-5}{4}$ And check the lines are parallel or perpendiclar.

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24 The Plane

Equation of a plane in normal form

1. **Vector form** the equation of a plane in normal form is given by $\vec{r}.\hat{n} = d$ where is a unit vector normal to the plane and d is distance of plane from the origin.
2. **Cartesian form** the equation of a plane is given by $lx + my + nz = p$ .where l, m, n are direction cosine of normal to the plane and p is a distance of a plane from origin.
3. Every equation of the form $\vec{r}.\hat{n} = d$ or $\vec{r}.\left(a\hat{i} + b\hat{j} + c\hat{k}\right) = d$ or $ax + by + cz = d$ always represent a plane where a, b, c are the directions ratios of the normal to this plane.

Equation of a plane perpendicular to a given vector and passing through a given point

1. **Vector form** $\left(\vec{r} - \vec{a}\right).\hat{n} = 0$ or $\vec{r}.\vec{n} = d$ where $d = \vec{a}.\vec{n}$ and $\vec{a}$ is a position vector.
2. **Cartesian form** Equation of plane passing through point (x_1, y_1, z_1)is given by a(x-x_1) + b(y-y_1) +c(z-z_1) = 0
   where a, b, c are the direction ratios of normal to the plane.

Equation of plane passing through three non collinear points

1. **Vector form** $\left(\vec{r} - \vec{a}\right).\left\{\left(\vec{b} - \vec{a}\right) \times \left(\vec{c} - \vec{a}\right)\right\} = 0$ where $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of three given points.
2. **Cartesian form If** (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3), are three non collinear points , then equation of plane is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$ If above points are collinear,

then $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

Equation of plane in intercept form

If a, b, and c are x intercept , y intercept and z intercept , respectively then equation of plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Equation of plane passing through the line of intersection of two given planes

1. **Vector form** : If equation of the plane are $\vec{r}.\vec{n_1}=d_1$ and $\vec{r}.\vec{n_2}=d_2$, then equation of plane is $\vec{r}.\left(\vec{n_1}+\lambda\vec{n_2}\right)=d_1+\lambda d_2$ where λ is a constant and calculated from given condition.
2. **Cartesian form :** If The equation of a plane are $a_1x+b_1y+c_1z=d_1$ and $a_2x+b_2y+c_2z=d_2$ then the equation of plane passing through the intersection of given planes is

   $\left(a_1x+b_1y+c_1z-d_1\right)+\lambda\left(a_2x+b_2y+c_2z-d_2\right)=0$

   where λ is a constant and calculated from given condition.

Equation of plane parallel to a given plane

1. **Vector form :** the equation of the plane parallel to the given plane $\vec{r}.\vec{n}=d_1$ is $\vec{r}.\vec{n}=d_2$ where d_2 is a constant.
2. **Cartesian form :** the equation of the plane parallel to the given plane $ax+by+cz+d_1=0$ is $ax+by+cz+d_2=0$.

Condition for coplanarity of two lines

1. **Vector form :** Two lines $\vec{r}=\left(\vec{a_1}+\lambda\vec{b_1}\right)$ and $\vec{r}=\left(\vec{a_2}+\mu\vec{b_2}\right)$ are coplanar, if $\left(\vec{a_2}-\vec{a_1}\right).\left(\vec{b_1}\times\vec{b_2}\right)=0$.

2. **Cartesian form :** Two lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and

$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ are coplanar, if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0 .$$

Angle between two planes

1. **Vector form** Two lines $\vec{n_1}=$ and $\vec{n_2}$ are normal to the plane $\vec{r}.\vec{n_1}=d_1$ and $\vec{r}.\vec{n_2}=d_2$ then angle between the normal to the plane is $\cos\theta=\left|\frac{\vec{n_1}.\vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|}\right|$.

(i) The planes are perpendicular to each other , if $\vec{n_1}.\vec{n_2}=0$.

(ii) The planes are parallel to each other $\vec{n_1}.=\lambda\vec{n_2}$ where λ is a scalar.

2. **Cartesian form** If two planes are $a_1x+b_1y+c_1z=d_1$ is $a_2x+b_2y+c_2z=d_2$ then $\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

(i) The planes are perpendicular to each other, if $a_1a_2+b_1b_2+c_1c_2$.

(ii) The planes are parallel to each other $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$.

Distance of a point from a plane

1. **Vector form :** The distance of a point whose position vector is $\vec{a}$ from the plane $\vec{r}.\hat{n} = d$ is $|\vec{a}.\hat{n} - d|$

(i) If the plane $\vec{r}.\hat{n} = d$, then the perpendicular distance is $\frac{|\vec{a}.\hat{n} - d|}{|\vec{n}|}$.

(ii) Length of perpendicular to the plane $\vec{r}.\hat{n} = d$ from the origin is $\frac{|d|}{|\vec{n}|}\left[\vec{a} = 0\right]$.

2. **Cartesian form** the distance of the point (x_1, y_1, z_1) from the plane Ax + By + Cz = D is $d = \left|\frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}}\right|$.

Angle between a line and a plane

1. **Vector form** If the equation of a line is $\vec{r} = \left(\vec{a} + \lambda\vec{b}\right)$ and the equation of the plane is $\vec{r}.\hat{n} = d$ then the angle $\cos\theta = \left|\frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}\right|$ or

If $\sin\phi = \sin(90 - \theta) = \left|\frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}\right|$.

2. **Cartesian form** If a1, b1, and c1 are the DR's of line and a2x + b2y + c2z +d = 0 is the equation of the plane , then $\sin\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$.

Practice problem

1) Find the equation of the plane passing through the following points:

a. (2, 1 ; 0), (3, -2, -2) and (3, 1, 7)

b. (0, -1, 0), (3, 3, 0) and (1, 1, 1)

c. (2,3,4) , (-3,5,1), (4,-1,2)

2) If the line drawn from the point (-2, -1, -3) meets a plane at right angle at the point (1, -3, 3), find the equation of the plane.

3) Find the equation of the plane which bisects the line segment joining the points A (2, 3, 4) and B (4, 5, 8) at right angles.

4) Find the vector and Cartesian Find the vector and Cartesian equation of the plane which passes through the point (5, 2, -4) and perpendicular to the line with direction ratios 2, 3, -1.

5) If O be the origin and coordinates of P be (1, 2, -3), then find the equation of the plane passing through P and perpendicular to OP.

6) If O is the origin and the coordinates of A are (a, b, c). Find the direction cosines of O A and the equation of the plane through A at right angles to OA.

7) Find the equation of the plane passing through the point (1, 2, 10) and perpendicular to the line joining the points (1, 4, 20)and (2, 3, 5). Find also the perpendicular distance of the origin from this plane.

8) Find the vector equation of the plane which is at a distance of $\frac{6}{\sqrt{26}}$ from the origin and its normal vector from the origin is $2\hat{i}-3\hat{j}+4\hat{k}$ Also, find its Cartesian from.

9) Find the distance of the plane 2x – 3y + 4z – 6 = 0 from the origin.

10) Find the vector equation of the plane passing through the points P (2, 5, -3), Q (-2, -3, 5) and R (5, 3, -3).

11) Find the vector equation of the plane passing through the points (1, 1, -1), (6, 4, -5) and (-4, -2, 3).

12) Find the equation of the plane through the points (2, 1, -1) and (-1, 3, 4) and perpendicular to the plane x - 2y + 4z = 10. Also, show that the plain thus obtained contains the line $\vec{r} = -\hat{i} + 3\hat{j} + 4\hat{k} + \lambda\left(3\hat{i} - 2\hat{j} - 5\hat{k}\right)$

13) Find the equation of the plane through the points (3, 4, 2) and (7, 0, 6) and is perpendicular to the plane 2x-5y=15.

14) Find the angle between the planes: 2x + y - 2z = 5 and 3x - 6y -z = 7

15) Find the equation of a plane passing through the point (-1, -1, 2) and perpendicular to the planes 3x + 2y – 3z = 1 and 5x – 4y + z = 5.

16) Obtain the equation of the plane passing through the point (1, -3, -2) and perpendicular to the planes x+2y+2z =5 and 3x + 3y + 2z = 8.

17) Find the equation of the plane passing through the points (1, -1, 2) and (2, -2, 2) and which is perpendicular to the plane 6x - 2y + 2z =9.

18) Find the equation of the plane with intercept 3 on the y-axis and parallel to ZOX plane.

19) Find the equation of the plane that contains the point (1, -1, 2) and is perpendicular to each of the planes 2x + 3y – 2z = 5 and x + 2y – 3z = 8.

20) Find the equation of the plane passing through (a, b, c) and parallel to the plane $\vec{r}.\left(\hat{i} + \hat{j} + \hat{k}\right) = 2$.

21) Find the equation of the plane passing through the point (-1, 3, 2) and perpendicular to each of the planes x + 2y + 3z = 5 and 3x + 3y + z = 0.

22) Find the vector equation of the plane through the points (2, 1, -1) and (-1, 3, 4) and perpendicular to the plane x – 2y + 4z = 10.

23) Find the equation of the plane which is perpendicular to the plane 5x + 3y + 6z + 8 = 0 and which contains the line of intersection of the planes x + 2y + 3z -4 = 0 and 2x + y - z + 5 = 0.

24) Find the cartesian as well as vector equations of the planes through the intersection of the planes $\vec{r}\left(2\hat{i}+6\hat{j}\right)+12=0$ and $\vec{r}\left(3\hat{i}-\hat{j}+4\hat{k}\right)=0$ which are at a unit distance from the origin.

25) Find the equation of the plane which contains the line of intersection of the planes x + 2y + 3z – 4 = 0 and 2x + y – z + 5 = 0 and which is perpendicular to the plane 5x + 3y – 6z + 8 =0.

26) Find the equation of the plane through the line of intersection of the planes $\vec{r}.\left(\hat{i}+3\hat{j}\right)+6=0$ and $\vec{r}.\left(3\hat{i}-\hat{j}-4\hat{k}\right)=0$, which is at a unit distance from the origin.

27) Find the equation of the plane that contains the line of intersection of the planes $\vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4=0$ and $\vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)+5=0$ which is perpendicular to the plane $\vec{r}.\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0$

28) Find the vector equation of the plane passing through the intersection of the planes $\vec{r}.\left(\hat{i}+\hat{j}+\hat{k}\right)=6$ and $\vec{r}.\left(2\hat{i}+3\hat{j}+4\hat{k}\right)=-5$ the point (1, 1, 1).

29) Find the equation of the plane passing through the intersection of the planes $\vec{r}.\left(2\hat{i}+\hat{j}+3\hat{k}\right)=7$, $\vec{r}.\left(2\hat{i}+5\hat{j}+3\hat{k}\right)=9$,and the point (2, 1, 3).

30) Find the equation of the plane through the intersection of the planes 3x – y + 2z = 4 and x + y + z = 2 and the point (2, 2, 1).

31) Find the vector equation of the plane through the line of intersection of the planes x + y + z = 1 and 2x+3y+4z=5 which is perpendicular to the plane x – y + z = 0.

32) Show that the points $\hat{i}-\hat{j}+3\hat{k}$ and $3\left(\hat{i}+\hat{j}+\hat{k}\right)$ are equidistant from the plane $\vec{r}.\left(5\hat{i}+2\hat{j}+\hat{k}\right)+9=0$ and lie on opposite sides of it.

33) If the points $(1,1,\lambda)$ and (-3, 0, 1) be equidistant from the plane $\vec{r}.\left(3\hat{i}+4\hat{j}-12\hat{k}\right)+13=0$ find the value of λ.

34) Find the distance between the point P (6, 5, 9) and the place determined by the points A(3, -1, 2), B (5,2, 4) and C(-1, -1, 6).

35) Find the equation of a plane passing through the point P (6, 5, 9) and parallel to the plane determined by the points A(3, -1, 2), B(5, 2, 4) and C(-1, -1, 6). Also, find the distance of this plane from the point A.

36) Two systems of rectangular axes have the same origin. If a plane cuts them to distances a, b, c and a', b', c' respectively, prove that $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}$

37) Find the distance between the point (7, 2, 4) and the plane determined by the points A(2, 5, -3), B(-2,-3, 5) and (5, 3, -3).

38) A plane makes intercepts -6, 3, 4 respectively on the coordinate axes. Find the length of the perpendicular from the origin on it.

39) Show that the line whose vector equation is $\vec{r}.\left(2\hat{i}-2\hat{j}+3\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}+4\hat{k}\right)$ is parallel to the plane whose vector equation is $\vec{r}\left(\hat{i}-5\hat{j}+\hat{k}\right)=5$. Also, find the distance between them.

40) Find the equation of the line passing though the point (3, 0, 1) and parallel to the planes x+2y=0 and 3y-z=0.

41) Find the equation of the plane passing through the line of intersection of the planes 2x+y–z=3,5x-3y+4z+9 = 0 and parallel to the $\frac{x-1}{2}=\frac{y-3}{5}+\frac{z-5}{5}$.

42) Show that the line whose vector equation is $\vec{r}=\left(2\hat{i}-2\hat{j}+3\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}+4\hat{k}\right)$ is parallel to the plane whose vector equation is $\vec{r}.\left(\hat{i}+5\hat{j}+\hat{k}\right)=5$. Also, find the distance between them.

43) Find the equations of the line passing through thee point (3, 0, 1) and parallel to the planes x + 2y = 0 and 3y – z = 0.

44) Find the equation of the plane passing through the line of intersection of the planes 2x+y –z=3 and 5x-3y+4z+9=0 and parallel to the line $\frac{x-1}{2}=\frac{y-3}{4}=\frac{z-5}{5}$.

45) Find the equation of the plane passing through the intersection of the planes $\vec{r}.\left(\hat{i}+\hat{j}+\hat{k}\right)=1$ and $\vec{r}.\left(2\hat{i}+3\hat{j}-\hat{k}\right)+4=0$ parallel to x-axis.

46) Find the equation of the plane passing through the point A(1, 2, 1) and perpendicular to the line joining the points P(1, 4, 2) and Q(2, 3, 5). Also, find the distance of this plane from the line $\frac{x+3}{2}=\frac{y-5}{-1}=\frac{z-7}{-1}$.

47) Find the equation of a plane passing through the points (0,0,0) and (3, -1, 2) and parallel to the line $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$.

48) Find the vector and cartesian equations of the line passing through (1, 2, 3) and parallel to the planes $\vec{r}.\left(\hat{i}-\hat{j}+2\hat{k}\right)=5$ and $\vec{r}.\left(3\hat{i}+\hat{j}+3\hat{k}\right)=6$.

49) Find the equation of the plane passing through the intersection of the planes x – 2y + z = 1 and 2x + y +z =8 and parallel to the line with direction ratios proportional to 1,2,1. Find also the perpendicular distance of (1,1,1) from this plane.

50) Find the vector and cartesian forms of the equation of the plane passing through the point (1, 2, -4) and parallel to the lines $\vec{r}=\left(\hat{i}+2\hat{j}-4\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$ and $\vec{r}=\left(\hat{i}-3\hat{j}+5\hat{k}\right)+\mu\left(\hat{i}+\hat{j}-\hat{k}\right)$ Also, find the distance of the point (9, -8, -10) from the plane thus obtained.

51) Find the equation of the plane passing through the points (3, 4, 1) and (0, 1, 0) and parallel to the line $\frac{x+3}{2}=\frac{y-3}{7}=\frac{z-1}{5}$.

52) Find the coordinates of the point where the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2}$ intersects the place $x-y+z-5=0$. Also, find the angle between the line and the place.

53) Find the vector equation of the line passing through (1, 2, 3) and perpendicular to the place $\vec{r}.\left(\hat{i}+2\hat{j}-5\hat{k}\right)+9=0$.

54) Find the angle between the line $\frac{x+1}{2}=\frac{y}{3}=\frac{z-3}{6}$ and the plane $10x+2y-11z=3$.

55) Find the vector equation of the line passing through (1, 2, 3) and parallel to the planes $\vec{r}.\left(\hat{i}-\hat{j}+2\hat{k}\right)=5$ and $\vec{r}.\left(3\hat{i}+\hat{j}+\hat{k}\right)=6$.

56) Find the value of λ such that the line $\frac{x-2}{6}=\frac{y-1}{\lambda}=\frac{z+5}{-4}$ is perpendicular to the place $3x-y-2z=7$.

57) Find the equation of the place passing through the points (-1, 2, 0), (2, 2, -1) and parallel to the line $\frac{x-1}{1}=\frac{2y-1}{2}=\frac{z+1}{-1}$.

58) Find the coordinates of the point where the line through the points A (3, 4, 1) and B (5, 1, 6) crosses the XY-plane.

59) Find the distance of the point (1, -2, 3) from the plane x – y + z = 5 measured parallel to the line whose direction cosines are proportional to 2, 3, -6.

60) Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses to

a. yz – plane **b.** zx – plane.

61) Find the coordinates of the point where the line through (3, -4, -5) and (2, -3, 1) crosses the plane.

62) Find the distance of the point (-1, -5, -10) from the point of intersection of the line $\vec{r}.\left(2\hat{i}-\hat{j}+2\hat{k}\right)+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right)$ and the plane $\vec{r}\left(\hat{i}-\hat{j}+\hat{k}\right)=5$.

63) Find the distance of the point (2, 12, 5) from the point of intersection of the line $\vec{r}.2\hat{i}-4\hat{j}+2\hat{k}+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right)$ and the plane $\vec{r}.\left(\hat{i}-2\hat{j}+\hat{k}\right)=0$.

64) Find the distance of the point P (-1, -5, -10) from the point of intersection of the line joining the points A (2, -1, 2) and B (3, 4, 4) with the plane x – y + z = 5.

65) Find the distance of the point P (3, 4, 4) from the point where the line joining the points A (-3, -4, -5) and B (2, -3, 1) intersects the plane 2x + y + z =7.

66) Show that the lines $\frac{x-a+d}{\alpha-\delta}=\frac{y-a}{a}=\frac{z-a-d}{\alpha+\delta}$ and $\frac{x-b+c}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-b-c}{\beta+\gamma}$.

67) If this lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y+k}{2}=\frac{z}{1}$ intersect, them find the value of k, and hence find the equation of the plane containing these lines.

68) Find the vector equation of the plane passing through the point (3, 4, 2) and (7, 0, 6) and perpendicular to the plane $2x-5y-15=0$. Also, show that the plane thus obtained contains the line $\vec{r}.\hat{i}+3\hat{j}-2\hat{k}+\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$.

69) If the lines $\frac{x-1}{-3}=\frac{y-2}{-2k}=\frac{z-3}{2}$ and $\frac{x-1}{k}=\frac{y-2}{1}=\frac{z-3}{5}$ are perpendicular, find the value of k and hence find the equation of the plane containing these lines.

70) Find the vector equation of the plane passing through three points with position vectors $\hat{i}+\hat{j}-2\hat{k}$,, $2\hat{i}-\hat{j}+\hat{k}$ and Also, find the coordinates of the point of intersection of this plane and the line $\hat{i}+2\hat{j}+\hat{k}.\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$.

71) Show that the lines $\frac{5-x}{-4}=\frac{y-7}{4}=\frac{z+3}{-5}$ and $\frac{x-8}{7}=\frac{2y-8}{2}=\frac{z-5}{3}$ are coplanar.

72) Find the equation of a plane which passes through the point (3, 2, 0) and contains the line $\frac{x-3}{1}=\frac{y-6}{5}=\frac{z-4}{4}$.

73) Find the length and the foot of the perpendicular from the point (7, 14, 5) to the plane 2x + 4y – z = 2. Also, find the image of the point P in the plane.

74) Find the image of the point having position vector $\hat{i}+3\hat{j}+4\hat{k}$ in the plane $\vec{r}.\left(2\hat{i}-\hat{j}+\hat{k}\right)+3=0$.

75) Find the distance of the point (1, -2, 3) from the plane x – y + z = 5 measured along a line parallel to $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$.

76) Find the distance of the point with position vector $-\hat{i}-5\hat{j}-10\hat{k}$ from the point of intersection of the line $\vec{r}.\left(2\hat{i}-\hat{j}+2\hat{k}\right)+\lambda\left(3\hat{i}+4\hat{j}+12\hat{k}\right)$ with the plane $\vec{r}.\left(\hat{i}-\hat{j}+\hat{k}\right)=5$.

77) Find the length and the foot of the perpendicular from the point (1, 1, 2) to the plane $\vec{r}.\left(\hat{i}-2\hat{j}+4\hat{k}\right)+5=0$.

78) Find the coordinates of the foot of the perpendicular and the perpendicular distance of the point P (3,2,1) from the plane . Find also image of the point in the plane.

79) Find the direction consines of the unit vector perpendicular to the plane $\vec{r}.\left(6\hat{i}-3\hat{j}-2\hat{k}\right)+1=0$ passing through the origin.

80) Find the coordinates of the foot of the perpendicular drawn from the origin to the plane 2x – 3y + 4z–6= 0.

81) Find the length and the foot of perpendicular from the point (1, 3/2, 2) to the plane 2x-2y+4z+5 = 0.

82) Write the angle between the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z+3}{-2}$ and the plane x + y + 4 = 0.

83) Write the intercept cut off by the plane 2x + y – z = 5 on x – axis.

84) Find the length of the perpendicular drawn from the origin to the plane $2x - 3y + 6z + 21 = 0$.

85) Write the vector equation of the line passing through the point (1, -2, -3) and normal to the plane $\vec{r}.\left(2\hat{i} + \hat{j} + 2\hat{k}\right) = 5$.

86) Write the vector equation of the plane, passing through the point (a, b, c) and parallel to the plane $\vec{r}.\left(\hat{i} + \hat{j} + \hat{k}\right) = 2$.

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25 Linear Programming

Some Defination and Result

In this section , we shall formally define various terms used in a linear programming problem.

The general form of a linear programming problem is

Optimize(Maximize or Minimize) $z = c_1x_1 + c_2x_2 + c_3x_3 + ...c_n x_n$

Subject to ,

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}\, x_n\, (\leq, =, \geq)\, b_{1.}.$

$a_{12}x_1 + a_{22}x_2 + a_{33}x_3 + ... + a_{2n}\, x_n\, (\leq, =, \geq)\, b_{2.}$

$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n (\leq, =, \geq) b_{m.}$

$x_1, x_2, x_3, ...x_n \geq 0$

Objective function

If $c_1, c_2, c_3, ...c_n$ are constant and $x_1, x_2, x_3, ...x_n$ are variables , then the linear function $Z = c_1x_1, c_2x_2, c_3x_3, ... + c_n\, x_n$ which is to be maximized or minimized is called the objective function.

Constraints

The inequations or equations in the variable of a L.P.P. which described the conditions under which the optimization (max. or mini.) is to be accomplished are called constraints. In the constraints given in the general form of a LPP there may be any one of the three signs. $(\leq, =, \geq)$

Non-negativity Restrictions

These are constraints which described the variables involved in a LPP are non-negative.

Feasible solution : A set of values of the variables $x_1, x_2, x_3, ...x_n$ is called a feasible solution of a LPP, if it satisfies the constraints and non negativity restriction of the problems.

Infeasible solution: A solution of a LPP is an infeasible solution , if it does not satisfy the non-negativity restrictions.

Feasible region : The common region determined by all the constraints of a LPP is called the feasible region and every point in this region is a feasible solution of the given LPP.

Optimal feasible solution : A feasible solution of a LPP is said to be an optimal feasible colution , if it also optimizes (maximize. or minimizes) the objective function.

Convex set : A set is a convex set, if every point on the line segment joining any two points in it lies in it.

Corner point : A corner point of afeasible region is a point in the region which is the intersection of two boundary lines.

Mathematical formulation of linear programming problems

Algorithm

Step1. In every LPP certain decisions are to be made . these decision variables are those quantities whose values are to be determined. Identify the variable and denote them by

$x_1, x_2, x_3.......$

Step2. Identify the objective function and express it as a linear function of the variables introduced in step1.

Step3. In a LPP the objective function may be in the form of max. profit or min. cost so, after expressing the objective function as a linear function of the decision variables,

Step4. Identify the set of constraints , stated in terms of decision variable and express them as linear inequations. Or equations.

Graphical method to solve lpp

Algorithm

Step1. First , write the given LPP in mathematical form by using mathematical formulation.

Step2. Consider all constraints as linear equation.

Step3. Draw the graph of each linear equations and find their intersection point.

Step4. Find the feasible region of the LPP and determine its corner points .

Step5. Evaluate the objective function Z = ax + by at each corner point Let M and m be, respectively denote the largest and smallest values of these points.

Step6. If the feasible region is bounded , then M and m are the max. and min. values of the objective function at corner points.If the feasible region is unbounded , then

(a) M is the max. value of objective function Z, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region .Otherwise , Z has no maximum value.

(b) m is the minimum value of Z if the open half plane determined by $ax + by < M$ has no point in common with the feasible region . Otherwise , Z has no minimum value.

Different type of linear programming problems:

Diet Problems

1) A dietician whishes to mix two types of food in such a way that the vitamin contents of the mixture contain at least 8 units of Vitamin A and 10 unit of vitamin C. Food 'I' contains 2 units per kg of vitamin A and 1 unit per kg of vitamin C while food 'II' contains 1 unit per kg of vitamin A and 2 units per kg of vitamin C. It costs Rs. 50.00 per kg to purchase food 'I' and Rs. 70.00 per kg to produce food 'II'. Formulate the above

linear programming problem to minimize the cost of such a mixture.

2) Solve each of the following linear programming problems by graphical method.

(i) Minimize $Z = 30x + 20y$

Subject to $x - y \geq 0$

$-x + 2y \geq 2$

$x \geq 3$

$y \leq 4$

$x, y \geq 0$

(ii) Minimize $Z = 3x_1 + 5x_2$

$x_1 + 3x_2 \geq 3$

$x_1 + x_2 \geq 2$

$x_1, x_2 \geq 0$

3) Solved the following linear programming problem graphically:
Maximize Z = 60x + 15y
Subject to constraints

$x + y \leq 50$

$3x + y \leq 90$

$x, y \geq 0$

4) Find graphically, the maximum value of $z = 2x + 5y$ subject to constraints given below:

$2x + 4y \leq 8$

$3x + y \leq 6$

$x + y \leq 4$

$x \geq 0, y \geq 0$

5) A diet for a sick person must contain at least 4000 units of vitamins, 50 units of minerals and 1400 of calories. Two foods A and B, are available at a cost of Rs. 4 and Rs. 3 per unit respectively. If one unit of A contains 200 units of vitamin, 1 unit of mineral and 40 calories and one unit of food B contains 100 units of vitamin, 2 units of minerals and 40 calories, find what combinations of foods should be used to have the least cost?

6) A diet is to contain at least 80 units of vitamin A and 100 units of minerals. Two foods F_1 and F_2 are available. Food F_1 costs Rs. 4 per unit and F_2 costs Rs. 6 per unit one unit of food contains 3 units of vitamin A and 4 units of minerals. One food contains F_2 6 units of vitamin A and 3 units of minerals. Formulate this as a linear programming problem and find graphically the minimum cost for diet that consists of mixture of these foods and also meets the mineral nutritional requirements.

7) Kellogg is a new cereal formed of a mixture of bran and rice that contains at least 88 grams of protein and 36 milligrams of iron. Knowing that bran contains 80 grams of protein and 40 milligrams of iron per kilogram, and that rice contains 100 grams of protein and 30 milligrams of iron per kilogram, find the minimum cost of producing this new cereal if bran costs Rs. 5 per kg and rice costs Rs. 4 per kg.

8) One kind of cake requires 300 gm of flour and 15 gm of fat, another king of cake requires 150 gm of flour and 30 gm of fat. Find the maximum number of cakes which can be made from 7.5 kg of flour and 600 gm of fat, assuming that there is no shortage of the other ingredients used in making the cakes. Make it as an LPP and solve it graphically.

Optimal product line problems

9) A manufacturer produces nuts and bolts for industrial machinery. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts while it takes 3 hours on machine

A and 1 hour on machine B to produce a package of bolts. He earns a profit of Rs. 2.50 per package of nuts and Re 1.00 per package of bolts. How many package of each should he produce each day so as to maximize his profit, if he operates his machines for at most 12 hours a day? Formulate this mathematically and then solve it.

10) An oil company requires 12,000, 20,000 and 15,000 barrels of high-grade, medium grade and low grade oil, respectively. Refinery A produces 100, 300 and 200 barrels per day of high-grade, medium-grade and low-grade oil, respectively, while refinery B produces 200, 400 and 100 barrels per day of high-grade, medium-grade and low-grade oil, respectively. If refinery A costs Rs. 400 per day and refinery B costs Rs. 300 per day to operate, how many days should each be run to minimize costs while satisfying requirements.

11) A company produces soft drink that has a contract which requires that a minimum of 80 units of the chemical A and 60 units of the chemical B to go into each bottle of the drink. The chemicals are available in a prepared mix from two different suppliers. Supplier S has a mix of 4 units of A and 2 units of B that costs Rs. 10, the supplier T has a mix of 1 unit of A and 1 unit of B that costs Rs. 4. How many mixes from S and T should the company purchase to honour contract requirement and yet minimize cost?

12) A dealer wishes to purchase a number of fans and sewing machines. He has only Rs. 5760.00 to invest and has space for at most 20 items. A fan costs him Rs. 360.00 and sewing machine Rs. 240.00. His expectation is that he can sell a fan at a profit of Rs. 22.00 and a sewing machine at a profit of Rs. 18.00. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit? Translate this problem mathematically and then solve it.

13) If a young man drives his scooter at a speed of 25 km/hr, he has to spend Rs. 2 per km on petrol. If he drives the scooter at a speed of 40 km/hour, it produces air pollution and increases his expenditure on petrol to Rs. 5 per km. He has a maximum of Rs. 100 to spend on petrol and travel a maximum distance in one hour time with less pollution. Express this problem as an LPP and solve it graphically. What value do you find here?

14) A factory manufactures two types of screws, A and B, each type requiring the use of two machines – an automatic and a hand-operated. It takes 4 minute on the automatic and 6 minutes on the hand-operated machines to manufacture a package of screws 'A', while it takes 6 minutes on the automatic and 3 minutes on the hand-operated machine to manufacture a package of screws 'B'. Each machine is available for at most 4 hours on any day. The manufacturer can sell a package of screws 'A' at a profit of 70 P and screw 'B' at a profit of Rs. 1. Assuming that he can sell all the screws he can manufacture, how many packages of each type should the factory owner produce in a day in order to maximize his profit? Determine the maximum profit?

15) A factory owner purchases two types of machines, A and B, for his factory. The requirements and limitations for machines are as follows:

Area occupied by the machine	**Labour force for each machine Daily**	**output in units**
Machine A	1000 sq. m	12 men 60
Machine B	1200 sq. m	08 men 40

He has an area of 7600 sq. m available and 72 skilled men who can operate the machines. How many machines of each type should he buy to maximize the daily output?

16) A manufacturer makes two types A and B of tea-cups. Three machines are needed for the manufacture and the time in minutes

required for each cup of the machines is given below:

Machines

	I	**II**	**III**
A	12	18	6
B	06	00	9

Each machine is available for a maximum of 6 hours per day. If the profit on each cup A is 75 paise and that on paise, show that 15 tea-cups of type A and 30 of type B should be manufactured in a day to get the maximum profit.

17) A company manufactures two types of novelty Souvenirs made of plywood. Souvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours 20 minutes available for cutting and 4 hours available for assembling. The profit is 50 paise each for type A and 60 paise each for type B souvenirs. How many souvenirs of each type should the company manufacture in order to maximize the profit?

18) A manufacturer produces two types of steel trunks. He has two machines A and B. For completing, the first types of the trunk requires 3 hours on machine A and 3 hours on machine B, whereas the second type of the trunk requires 3 hours on machine A and 2 hours on machine B. Machine A and B can work at most for 18 hours and 15 hours per day respectively. He earns a profit of Rs. 30 and Rs. 25 per trunk of the first type and the second type respectively. How many trunks of each type must he make each day to make maximum profit?

19) A gardener has supply of fertilizer of type I which consists of 10% nitrogen and 6% phosphoric acid and type II fertilizer which consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, he finds that he needs at least 14 kg of nitrogen and 14 kg of phosphoric acid for his crop. If the

type I fertilizer costs 60 paise per kg and type II fertilizer costs 40 paise per kg, determine how many kilograms of each fertilizer should be used so that nutrient requirements are met at a minimum cost. What is the minimum cost?

20) A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of grinding/cutting machine and a sprayer. It takes 2 hours on the grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp while it takes 1 hour on the grinding/cutting machine and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at most 20 hours and the grinding/cutting machine for at most 12 hours. The profit from the sale of a lamp is Rs. 5.00 and a shade is Rs. 3.00. Assuming that the manufacturer can sell all the lamps and shades that he produces, how should he sechdule his daily production in order to maximize his profit.

21) A producer has 30 and 17 units of labour and capital respectively which he can use to produce two type of goods X and Y. To produce one unit of X, 2 units of labour and 3 units of capital are required. Similarly, 3 units of labour and 1 unit of capital is required to produce one unit of Y. If X and Y are priced at Rs. 100 and Rs. 120 per unit respectively, how should be producer use his resources to maximize the total revenue? Solve the problem graphically.

22) A firm manufactures two types of products A and B and sells them at a profit of Rs. 5 per unit of type A and Rs. 3 per unit of type B. Each product is processed on two machines $M_1$and M_2. One Unit of type A requires one minute of processing time onM_1 and two minutes of processing time M_2, whereas one unit of type B requires one minute of processing time on M_1 and one minute on M_2. Machine and are respectively available for at most 5 hours and 6 hours in a day. Find out how many units of each type of product should the firm produce a day in order to maximize the profit. Solve the problem graphically.

23) A small firm manufactures items A and B. The total number of items A and B that it can manufacture in a day is at the most 24. Item A takes one hour to make while item B takes only half an hour. The maximum time available per day is 16 hours. If the profit on one unit of item A be Rs. 300 and one unit of item B be Rs. 160, how many of each type of item be produced to maximize the profit? Solve the problem graphically.

24) A company manufactures two types of toys A and B. Type A requires 5 minutes each for cutting and 10 minutes each for assembling. Type B required 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours available for cutting and 4 hours available for assembling in a day. The profit is Rs. 50 each on type A and Rs. 60 each on type B. How many toys of each should the company manufacture in day to maximize the profit?

25) A company manufactures two articles A and B. There are two departments through which these articles are processed: (i) assembly and (ii) finishing departments. The maximum capacity of the first department is 60 hours a week and that of other department is 48 hours per week. The product of each unit of article A requires 4 hours in assembly and 2 hours in finishing and that of each unit of B requires 2 hours in assembly and 4 hours in finishing. It the profit is Rs. 6 for each unit of A and Rs. 8 for each unit of B, find the number of units of A and B to be produced per week in order to have maximum profit.

26) A firm makes items A and B the total number of items it can make in a day is 24. It takes one hour to make an item of A and half an hour to make an item of B. The maximum time available per day is 16 hours. The profit on an item of A is Rs. 300 and on one item of B is Rs. 160. How many items of each type should be produced to maximize the profit? Solve the problem graphically.

27) If a young man drives his vehicle at 25 km/ hr, he has to spend Rs. 2 per km on petrol. If he drives it at a faster speed of 40 km/hr, the petrol cost increases to Rs. 5/per km. He has Rs. 100 to spend on petrol and travel within one hour. Express this as an LPP and solve the same.

28) An oil company has two depots, A and B, with capacities of 7000 litres and 4000 litres respectively. The company is to supply oil to three petrol pumps, D, E, f whose requirements are 4500, 3000 and 3500 litres respectively. The distance (in km) between the depots and petrol pumps is given in the following table:

To /From	Distance (in km)	
	A	**B**
D	7	3
E	6	4
F	3	2

Assuming that the transportation cost per km. is Rs. 1.00 per litre, how should the delivery be scheduled in order that the transportation cost is minimum?

29) A small firm manufactures gold rings and chains. The total number of rings and chains manufactured per day is 24. It takes 1 hour to make a ring and 30 minutes to make a chain. The maximum number of hours available per day is 16. If the profit on a ring is Rs. 300 and that on a chain is Rs. 190, find the number of rings and chains that should be manufactured per day, so as to earn the maximum profit. Make it as an LPP and solve it graphically.

30) A library has to accommodate two different types of books on a shelf. The books are 6 cm and 4 cm thick and weigh 1 kg and 1kg each respectively. The shelf is 96 cm long and atmost can support a weigh of 21 kg. How should the shelf be filled with

the books of two types in order to include the greatest number of books? Make it as an LPP and solve it graphically.

31) A factory makes tennis rackets and cricket bats. A tennis racket takes 1.5 hours of machine time and 3 hours of craftmen's time in its making while a cricket bat takes 3 hours of machine time and 1 hour of craftman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftman's time. If the profit on a racket and on a bat is Rs. 20 and Rs. 10 respectively, find the number of tennis rackets and cricket bats that the factory must manufacture to earn the maximum profit. Make it as an LPP and solve it graphically.

32) A merchant plans to sell two types of personal computers a desktop model and a portable model that will cost Rs. 25,000 and Rs. 40,000 respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than Rs. 70 lakhs and his profit on the desktop model is Rs. 4,500 and on the portable model is Rs. 5,000. Make an LPP and solve it graphically.

33) A cooperative society of farmers has 50 hectare of land to grow two crops X and Y. The profit from crops X and Y per hectare are estimated as Rs. 10,500 and Rs. 9,000 respectively. To control weeds, a liquid herbicide has to be used for crops X and Y at rates of 20 liters and 10 liters per hectare. Further, no more than 800 litres of herbicide should be used in order to protect fish and wild life using a pond which collects drainage from this land. How much land should be allocated to each crop so at to maximize the total profit of the society?

34) A manufacturer considers that men and women workers are equally efficient and so he pays them at the same rate. He has 30 and 17 units of workers (male and female) and capital

respectively, which he uses to produce two types of goods A and B. To produce one unit of A, 2 workers and 3 units of capital are required while 3 workers and 1 unit of capital is required to produce one unit of B. If A and B are priced at Rs. 100 and Rs. 120 per unit respectively, how should he use his resources to maximize the total revenue? From the above as an LLP and solve graphically. Do you agree with this view of the manufacturer that men and women workers are equally efficient and so should be paid at the same rate?

Transportation problems

35) Two godowns, A and B, have grain storage capacity of 100 quintals and 50 quintals respectively. They supply to 3 ration shops, D,E and F, whose requirements are 60, 50 and 40 quintals respectively. The cost of transportation per quintal from the godowns to the shops are given in the following table:

	Transportation cost per quintal (in Rs.)	
FromTo	**A**	**B**
D	6.00	4.00
E	3.00	2.00
F	2.50	3.00

How should the supplies be transported in order that the transportation cost is minimum?

——==00==——

26 Probability

Experiment and Events

Experiment

An operation which can produce some well defined outcomes is called an experiment.

Random experiment : An experiment is called random experiment, if it satisfies following conditions-

1) It has more than one possible conditions.
2) It is not possible to predict the outcomes in advance.

Sample space: The set of all possible outcomes of a random experiment called the sample space associated with it.

Trial :The number of times the experiment is repeated is called the number of trials.

Event A: Subset of the sample space associated with arandom experiment is called an event.

Sure Event: An event associated with a random experiment is called a certain or sure event if it always occur whenever the experiment is performed.

Impossible Event: An event associated with a random experiment is called an impossible event if it never occurs whenever the experiment is performed.

Compound Event: An event associated with a random experiment is called an compound event , if it is the disjoint union of two or more elementary events.

Mutually exclusive events: Two or more events associated with a random experiment are said to be mutually exclusive if no two or more of them can occur simultaneously in the same trial.

If A and B are mutually exclusive then $(A \cap B) = \varnothing$

Exhaustive events: Two or more events associated with a random experiment are said to be exhaustive if their union is the sample space.

Equally likely event: The given events are said to be equally likely if none of them is expected to occur in preference to the other .

e.g. In throwing an unbiased die , all the six faces are equally likely to come.

Complement of an event: let A be An event in a sample space S, then complement of A is the set of all sample point which are not in A and it is denoted by A' or $\overline{\overline{A}}$

PROBABILITY OF AN EVENT If there are an elementary event associated with a random experiment and m of them are favorable to an event A , then the probability of happening ofAis denoted by P(A) and is defined by $\frac{m}{n}$

Thus, $\mathrm{P}(A)=\frac{m}{n}=\frac{n(A)}{n(S)}$

If $\mathrm{P(A)=1}$, then A is called the certain event and A is called an impossible event If $\mathrm{P(A)=0}$

Also, $\mathrm{P(A)+P(\overline{A})}=1$

NOTE : (i) $0 \le P(A) \le 1$

(II) Probability of impossible event is 0.

(i) Probability of certain event is 1.

(ii) $\mathrm{P(A \cup A')=P(S)}$

(iii) $\mathrm{P(A)+P(A')=1}$

(iv) $\mathrm{P(A} \cap A') = P(\varnothing)$

(v) $\mathrm{P(A')'=P(A)}$

(vi) P(AUB)=P(A)+P(B)-P(A $\cap$ B)

Conditional probability

Let E and F be two events associated with the same sample space of a random experiment . Then , probability of occurrence of event E , when the event F is already occurred , is called conditional probability of event E over F and it is denoted by P(E\F) .

P(E\F)=$\dfrac{P(E \cap F)}{P(F)}$ where P(F) $\neq 0$

Similarly, conditional probability of event F over E is given by

P(F\E)=$\dfrac{P(F \cap E)}{P(E)}$ where P(E) $\neq 0$.

Propertikes of conditional probability

If E and F are two events of sample space S and G is an event of S which has already occurred such that

P(G) $\neq 0$, then

(i) P(EUF)/G=P(E/G)=P(F/G)-P(E $\cap$ F)/G

(ii)

, are mutually exclusive event.

(iii) P(E'/G)=1- P(E/G)I;

Type I. Problem based on

P(E/F)=$\dfrac{n(E \cap F)}{n(F)}, P(F / E) = \dfrac{n(F \cap E)}{n(E)}$

1) A fair date dice is rolled. Consider the following events:

$A = \{1,2,3\}, B = \{2,3\}, and\ C = \{2,3,4,5\}$

(i) $P(A / B)$ and P(B/A) (ii) P(A/C) and P(C/A)

(iii) P(A U B/C) and P(A $\cap$ B/C) (iv) P(A $\cap$ B/C)

2) A coin is tossed three times. Find P (E/F) in each of the following:
   (i) E = Head on the third toss, F = Heads of first two tosses
   (ii) E = At least two heads, F = At most two heads
   (iii) E = At most two tails, F = At least one tail
3) Two coins are tossed once. Find P(E/F) in each of the following:
   (i) E = Tail appear on one coin, F = One coin shows head
   (ii) E = No tail appears, F = No head appears.
4) Mother, father and son line up at random for a family picture. Find P(AB), if A and B are defined as follows:
   A = Son on one end, B = Father in the middle
5) A couple has two children. Find the probability that.
   (i) Both the children are boys, if it is known that the older child is a day.
   (ii) Both the children are girls, if it is known that the older child is a girl.
   (iii) Both the children are boys, if it is known that at least one of the children is a boy.
6) A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the probability that the number 4 has appeared at least once?
7) A die is thrown three times. Events A and B are defined as below: A = Getting 4 on third die, B = Getting 6 on the first and 5 on the second throw

Find the probability of A given that B has already occurred.

8) A black and a red dice are rolled in order. Find the conditional probability of obtaining
   (i) A sum greater than 9, given that the black die resulted in 5.
   (ii) A sum 8, given that the red die resulted in a number less than 4.

9) Consider the experiment of throwing a die, if a multiple of 3 comes up, threw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail' given 'at least one die shows a three'.

10) In a school, there are 1000 students, out of which 430 are girls, It is known that out of 430, 10% of the girls study in class XII. What is the probability that a student chosen randomly studies in class XII given that the chosen student is a girl?

11) An instructor has a question bank consisting of 300 easy True/false questions, 200 difficult True/False questions, 500 easy multiple choice question and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be any easy question given that it is a multiple choice question?

TYPE II- PROBLEM BASED ON

$$P(E/F) = \frac{P(E \cap F)}{P(F)}, P(F/E) = \frac{P(F \cap E)}{P(E)}$$

12) Given that A and B are two events such that

$P(A) = 0.6, P(B) = 0.3$ *and* $P(A \cap B) = 0.2$, find P(A/B) and P(B/A) .

13) If $P(A) = \frac{6}{11}, P(B) = \frac{5}{11}$ *and* $P(A \cup B) = \frac{7}{11}$, *find*

(i) $P(A \cap B)$, (ii) $P(A/B)$ (iii) $P(B/A)$

14) Evaluate $P(A \cup B)$, *if* $2P(A) = P(B)$ *and* $P(A/B) = \frac{2}{5}$.

15) Assume that each child born is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls given that (i) the youngest is a girl, (ii) at least one is a girl?

Multiplicaeion theorem on probability

If A and B are two events associated with a random experiment , then

$$P(A \cap B) = P(A)P(B / A), if \ \mathrm{P(A)} \neq 0 \quad \text{or}$$

$$P(A \cap B) = P(B)P(A / B), if \ \mathrm{P(B)} \neq 0$$

Particular case If A, B, C are three events associated with a random experiment , then $P(A \cap B \cap C) = P(A)P(B / A)P(C / A \cap B)$.

TYPE I on finding the probability of simultaneous occurrence of two or more events

16) A couple has two children. Find the probability that both the children are (i) males, if it is known that at least one of the children is male. (ii) females, if is known that the elder child is a female.

17) Find the probability of drawing diamond cash in each of the consecutive draws from a well shuffled pack of cards, if the card drawn is not replaced after the first draw.

18) From a pack of 52 cards, 4 are drawn one by one without replacement. Find the probability that all are aces (or, kings).

19) A bag contains 5 white, 7 red and 3 black balls. It three balls are drawn one by one without replacement. What is the probability that none is red.

20) An urn contains 10 black and 5 white balls. Two balls are drawn from the urn one after the other without replacement. What is the probability that both drawn balls are black?

21) Three cards are drawn successively, without replacement from a pack of 52 well shuffled cards. What is the probability that first two cards are kings and third card drawn is an ace?

22) Three events A, B and C have probabilities $\frac{2}{5}, \frac{1}{3} and \frac{1}{2}$ respectively. Given that

$P(A \cap C) = \frac{1}{5}$ and $P(B \cap C) = \frac{1}{4}$, find the values of P(C/B) and $P(\bar{A} \cap \bar{C})$.

23) 10% of the bulbs produced in a factory are red colour and 2% are red and defective. If one bulb is picked at random, determine the probability of its being defective if it is red.

24) A couple has 2 children. Find the probability that both are boys, if it is known that (i) one of the children is a boy (ii) the older child is a boy.

25) Consider a random experiment in which a coin is tossed and if the coin shows head it is tossed again but if it shows a tail then a die is tossed. If 8 possible outcomes are equally likely, find the probability that the die shows a number greater than 4 if it is known that the first throw of the coin results in a tail.

26) A committee of 4 students is selected at random from a group consisting of 8 boys and 4 girls. Given that there is at least one girl in the committee, calculate the probability that there are exactly 2 girls in the committee.

27) Three die is thrown three times. Events A and B are defined as follows:

A : 4 on the third throw, B : 6 on the first and 5 on)")" the second throw.

Find the probability of A given that B has already occurred.

28) Three dice are thrown at the same time. Find the probability of getting three two's if it is know that the sum of the numbers on the dice was a six.

Independent events

Two events are said to be independent , if the occurrence or nonoccurrence of one does not affect the probability of the occurrence or non – occurrence of the other. For any two independent events A & B we have the relation.

(i) $P(A \cap B) = P(A).P(B)$

(ii) $P(A / B) = P(A), P(B) \neq 0$

(iii) $P(B / A) = P(B), P(A) \neq 0$

(iv) Complement of independent events, $P(\overline{A} \cap \overline{B}) = P(\overline{A}).P(\overline{B})$

Practice problem related independent and dependent event

29) Consider the experiment of tossing a coin. If the coin shows head toss it again but if it shows tail then throw a die. Find the conditional probability of the event 'the die shows a number greater than 4, given that 'three is at least one tail'.

30) If A and B are two events such that (i) $P(A) = \frac{1}{2}, P$ (B) $= \frac{1}{3}$ *and* P(A $\cap$ B)$=\frac{1}{4}$, find P(A/B), P(B/A),P($\overline{A}$ /$\overline{B}$) and P ($\overline{A}$ /$\overline{B}$)

31) A die is thrown twice and the sum of the numbers appearing is observed to be 8. What is the conditional probability that the number 5 has appeared at least once?

32) Assume that each born child is equally likely to be a boy or girl. If a family has two children, what is the constitutional probability that both are girls? Given that

(i) The youngest is a girl (ii) at least one is girl.

33) If A and B are independent event associated with a random experiment, then prove that

(i) $\overline{A}$ and B are independent events

(ii) A and $\overline{B}$ are independent events

(iii) $\overline{A}$ and $\overline{B}$ are also independent events.

34) An unbiased die is thrown twice. Let the event A be 'odd number

on the first throw' and B the event 'odd number on the second throw'. Check the independence of events A and B.

35) Three coins are tossed. Consider the events: E = three heads or three tails, F = At least two heads and G = At most two heads. Of the pairs (E, F), (E, G) which are independent? Which are dependent?

36) A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event' 3 on the die.' .Check whether A and B are independent event or not.

37) A die is marked 1, 2, 3 in red and 4,5, 6 in green is tossed. Let A be the event' number is even' and B be the event' number is red'. Are A and B independent?

38) Events A and B are such that $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and P (not A or not B) $= \frac{1}{4}$, State whether A and B are independent?

39) A die is thrown once. If A is the event "the number appearing is a multiple of 3" and B is the event "the number appearing is even". Are the events A and B independent?

40) Two dice are thrown together. Let A be the event "getting 6 on the first die" and B be the event "getting 2 on the second die". Are the events A and B independent?

41) For a loaded die, the probabilities of outcomes are given as under:

$$P(1) = P(2) = \frac{2}{10} , P(3) = P(5) = P(6) = \frac{1}{10} \text{ and } P(4) = \frac{3}{10}$$

The die is thrown two times. Let A and B be the events as defined below

A = Getting same number each time, B = Getting a total score of 10 or more.

Determine whether or not A and B are independents events.

42) Given that the events A and B are such that $P(A) = \frac{1}{2}, P(A \cup B) = \frac{3}{5} and P(B) = P$. Find p, if they are (i) mutually exclusive, (ii) independent.

43) If A and B are two independent events such that $P(\overline{A} \cap B) = \frac{2}{15}$ *and* $P(A \cap \overline{B}) = \frac{1}{6}$, *then* find P(A) and p(B)

44) A and B are two candidates seeking admission in a college. The probability that A is selected is 0.7 and the probability that exactly one of them is selected is 0.6. Find the probability that B is selected.

45) Probability of solving a specific problem independently by A and B are $\frac{1}{2}$ *and* $\frac{1}{3}$ respectively. If both try by solve the problem independently, find the probability that

(i) The problem is solved (ii) exactly one of them solve the problem.

46) A can hit a target 4 times in 5 shots, B 3 times in 4 shots, and C 2 times in 3 shots. Calculate the probability that

(i) A, B, C all may hit (ii) B, C may hit and may not.

(iii) Any two of A, B and C will hit the target

(iv) None of them will hit the target.

47) A card is drawn from a pack of 52 cards so that each card is equally likely to be selected. In which of the following cases are the events A and B independent?

(i) A = the card drawn in a king or queen, B = the card drawn is a queen or jack

(ii) A = the card drawn is black, B = the card drawn is a king

(iii) B = the card drawn is a spade, B = the card drawn in an ace.

48) If $P(notB) = 0.65, P(A \cup B) = 0.85,$ and A and B are independent events, then find P (A).

49) If A and B are two independent events such that $P(\overline{A} \cap B) = 2/15$ and $P(A \cap \overline{B}) = 1/6, then$ find P(B).

50) Two balls are drawn at random with replacement from box containing 10 black and 8 red balls. Find the probability that (i) both balls are red, (ii) first ball is black and second is red, (iii) one of them is black and other is red.

51) An urn contains 4 red and 7 black balls. Two balls are drawn at random with replacement. Find the probability of getting (i) 2 red balls, (ii) 2 black balls, (iii) one red and one black ball.

52) The probability of two students A and B coming to the school in time are $\frac{3}{7} and \frac{5}{7}$ respectively. Assuming that the events, 'A coming in time' and 'B coming in time' are independent, find the probability of only one of them coming to the school in time. Write at least one advantage of coming to school in time.

53) Two dice are thrown together and total score is notes The event E, F and G are "a total 4", "a total 9 or more", and "a total divisible by 5", respectively. Calculate P(E), P(F) and P(G) and decide which pairs of events, if any, are independent.

54) A bag contains 5 red marbles and 3 black marbles. Three marbles are drawn one by one without replacement. What is the probability that at least one of the three marbles drawn be bloack, if the first marble is red?

55) A speaks in 60% of the cases and B in 90% of the cases. In what percentage of cases are they likely to (i) contradict each other in stating the same fact? (ii) agree in stating the same fact?

56) A and B thrown alternately a pair of dice. A wins if he throws 6 before B throws 7 and B wins if the throws 7 before A throws 6. Find their respective chance of winning, if A begins.

57) Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that (i) both the balls are red. (ii) first ball is black and second is red. (iii) one of them is black and other is red.

58) X is taking up subjects – Mathematics, Physics and Chemistry in the examination. His probabilities of getting grade A in these subjects are 0.2, 0.3 and 0.5 respectively. Find the probability that he gets.

(i) Grade A in all subject (ii) Grade A in no subject

(iii) Grade A in two subjects.

59) Out of 100 students, two sections of 40 and 60 are formed. If you and your friends are among 100 students, what is the probability that: (i) you both enter the same section? (ii) you both enter the different sections?

60) In a hockey match, both teams A and B scored same number of goals upto the end of the game, so to decide the winner, the refree asked both the captains to throw a die alternately and decide that the team, whose captain gets a first six, will be declared the winner. If the captain of team A was asked to start, find their respective probabilities of wining the match and state whether the decision of the refree was fair or not.

The law of total probability

Theorem: Let S be the sample space and let $E1, E2, E3$, …….. En be n mutually exclusive and exhaustive events associated with the random experiment . If A any Event occur with $E1, E2, E3$ …. En , then,

$$P(A) = P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + \dots\dots\dots\dots\dots P(E_n\ P(A/E_n\ )$$

Practice problem related total probability

61) One bag contains 4 white and 5 black balls. Another bag contains 6 white and 7 black balls. A ball is transferred from first bag to the second bag and then a ball is drawn from the second bag. Find the probability that the ball drawn is white.

62) There are two bags, one of which contains 3 black and 4 white, balls, white the other contains 4 black and 3 white balls. A fair die is cast, if the face 1 or 3 turns up, a ball is taken from the first bag, and if any other face turns up a ball is chosen from the second bag. Find the probability of choosing a black ball.

63) A bag A contains 4 black and 6 red balls and bag B contains 7 black and 3 red balls. A die is thrown. 1 or 2 appears on it, then bag A is chosen, otherwise bag B. if two balls are drawn at random (without replacement) from the selected bag, find the probability of one of them being red and another black.

64) A box has 5 blue and 4 red balls. One ball is drawn at random and not replaced. Its colour is also not noted. Then another ball is drawn at random. What is the probability of second ball being blue?

65) An urn contains m white and n black balls. A ball is drawn at random and is put back into the urn along with k balls of the same colour as that of the ball drawn. A ball is again drawn at random. Show that the probability of drawing a white ball now does not depend on k.

66) A bag contains $(2n+1)$ coins. It is known that n of these coins have a head on both sides whereas the rest of the coins are fair. A coin is picked up at random from the bag and is tossed. If the probability that the toss results in a head $\frac{31}{42}$, determine the value of n.

67) Three bags contains a number of red and white balls as follows: Bag I : 3 red balls ; Bag II : 2 red balls and 1 white ball ; Bag III : 3 white balls.

The probability that bag i will be chosen and a ball is selected from it is $\frac{i}{6}$, i = 1, 2, 3. What is the probability that.

(i) a red ball will be selected?

(ii) a white ball will be selected?

68) One bag contains 4 yellow and 5 red balls. Another bag contains 6 yellow and 3 red balls. A ball is transferred from the first bag to the second bag and then a ball is drawn from the second bag. Find the probability that ball drawn is yellow.

69) A bag contains 3 white and 2 black balls and another bag contains 2 white and 4 black balls. One bag is chosen at random. From the selected bag, one ball is drawn. Find the probability that the ball drawn is white.

70) The bag A contains 8 white and 7 black balls while the bag B contains 5 white and 4 black balls. One ball is randomly picked up from the bag A and mixed up with the balls in bag B. Then a ball is randomly drawn out from it. From the probability that ball drawn is white.

71) Three machines $E_1, E_2, E_3,$ in a certain factory produce 50%, 25% and 25% respectively, of the total daily output of electric bulbs. It is known that 4% of the tubes produced one each of machines E_1 and E_2 are defective, and that 5% of those produced on E_3 are defective. If one tube is picked up at random from a day's production, calculate the probability that it is defective.

Baye's theorem

Theorem: Let S be the sample space and $E_1, E_2, E_3,$ .... En be n mutually exclusive and exhaustive events associated with a random experiment . If A is any event which occurs with E_1 , or E_2 or ...... or E_n , then

$$P(E_i / A) = \frac{p(E_i)P(A / E_i)}{\sum_{i=1}^{n} P(E_i)P(A / E_i)'} I = 1, 2, 3, ..., n$$

Practice problem related baye's theorem

72) In a bolt factory, machine A, B and C manufacture respectively 25%, 35% and 40% of the total bolts. Of their output 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product. If the bolt drawn is found to be defective, what is the probability that it is manufactured by the machine B?

73) A company has two plants to manufacture scooters. Plant I manufactures 70% of the scooters and Plant II manufactures 30%. At Plant I, 80% of the scooters are rated as of standard quality and at Plant II, 90% of the scooters are rated as of standard quality. A scooter is chosen at random and is found to be of standard quality. What is the probability that it has come from Plant II?

74) An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probabilities of an accident involving a scooter driver, car driver and a truck driver are 0.01, 0.03 and 0.15 respectively. One of the insured person meets with an accident. What is the probability that he is a scooter driver?

75) A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be hearts. Find the probability of the missing card to be a heart.

76) Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets a 1, 2, 3 or 4 she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head; what is the probability that she threw a 1, 2, 3 or 4 with the die?

77) Given three identical boxes I, II and III, each containing two coins. In box I both coins are gold coins, in box II both are silver coins and in box III there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold?

78) Bag I contains 3 red and 4 black balls and Bag II contains 4 and 5 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.

79) A bag contains 4 balls. Two balls are drawn at random and are found to be white. What is the probability that all balls are white?

80) A bag contains 3 red and 7 black balls. Two balls are selected at random one-by-one without replacement. If the second selected ball happens to be red, what is the probability that the first selected ball is also red?

81) Suppose that 6% of the people with blood group O are left handed and 10% of those with other blood groups are left handed. 30% of the people have blood group O. If a left handed person is selected at random, what is the probability that he/she will have blood group O?

82) A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

83) In a test, an examine either guesses or copies or knows the answer to a multiple choice question with four choices. The

probability that he makes a guess is 1/3 and the probability that he copies the answer is 1/6. The probability that his answer is correct, given that he copied it, is 1/8. Find the probability that he knew the answer to question, given that he correctly answered it.

84) A letter is known to have come either from TATANAGAR or CALCUTTA. On the envelope just two consecutive letters TA are visible. What is the probability that the letter has come from (i) Calcutta (ii) Tatanagar?

85) A doctor is to visit a patient. From the past experience, it is know that the probabilities that he will come by train, bus, scooter or by other means of transport are respectively $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}, and \frac{2}{5}$. The probability that he will be late are $\frac{1}{4}, \frac{1}{3}, and \frac{1}{12}$, if he comes by train, bus and scooter respectively, but if he comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by train?

86) Suppose that he reliability of a HIV test is specified as follows: Of people having HIV, 90% of the test detect the disease but 10% go undetected. Of people free of HIV, 99% of the test are judged HIV-ive but 1% are diagnosed as showing HIV + ive. From a large population of which only 0.1% have HIV, one person is selected at random, given the HIV test, and the pathologist reports him/her as HIV + ive. What is the probability that the person actually has HIV?

87) Suppose you have two coins which appear identical in your pocket. You known that one is fair and one is 2-headed. If you take one out. Loss it and get head, what is the probability that it was a fair coin?

88) Three bags contain a number of red and white balls are as follows:

Bag I : 3 red balls; Bag II : 2 red balls and 1 white ball; Bag III: 3 white balls

The probability that bag i will be chosen and a ball is selected from it

is $\frac{i}{6}=, i=1,2,3$. If a white ball is selected, what is the probability that it came from (i) Bag II (ii) Bag III.

89) A shopkeeper sells three types of seeds A_1, A_2 *and* A_3 . They are sold as a mixture where the proportions are 4 : 4 : 2 respectively. The germinations rates of three types of seeds are 45%, 60% and 35%. Calculate the probability

(i) That is will not germinate given that the seed is of type

(ii) Of a randomly chosen seed to germinate.

(iii) That it is type given that a randomly chosen seed does not germinate.

90) The contents of urns I, II, III are as follows:

Urn I : 1 white, 2 black and 3 red balls

Urn II : 2 white, 1 black and 1 red balls

Urn III: 4 white, 5 black and 3 red balls.

One urn is chosen at random and two balls are drawn. They happen to be white and red. What is the probability that they come from Urns I, II, III?

91) A bag A contains 2 white and 3 red balls and a bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag B.

92) Three urns contain 2 white and 3 black balls; 3 white and 2 black balls and 4 white and 1 black ball respectively. One ball

is drawn from an urn chosen at random and it was found to be white. Find the probability that it was drawn from the first urn.

93) Suppose a girl throws a die. If she gets 1 or 2, she tosses a coin three times and notes the number of tails. If she gets 3, 4, 5 or 6, she tosses a coin once and notes whether a 'head' or 'tail' is obtained. If she obtained exactly one 'tail', what is the probability that she threw 3, 4, 5 or 6 with the die?

94) Two groups are competing for the positions of the Board of Directors of a Corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding probability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group.

95) A factory has three machines X, Y and Z producing 1000, 2000 and 3000 bolts per day respectively. The machine X produces 1% defective bolts, Y produces 1.5% and Z produces 2% defective bolts. At the end of a day, a bolt is drawn at random and is found to be defective. What is the probability that this defective bolt has been produced by machine X?

96) An insurance company insured 3000 scooters, 4000 cars and 5000 trucks. The probabilities of the accident involving a scooter, a car and a truck are 0.02, 0.03 and 0.04 respectively. One of the insured vehicles meet with an accident. Find the probability that it is a (a) scooter (ii) car (iii) truck.

97) Suppose we have four boxes A, B, C, D containing coloured marbles as given below:

Marble Colour Box	**Red**	**White**	**Black**
A	1	6	3
B	6	2	2
C	8	1	1
D	0	6	4

One of the boxes has been selected at random and a single marble is drawn from it. If the marble is red, what is the probability that it was drawn from box A? box B? box C?

98) A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, whereas the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B on the job for 30% of the time and C on the job for 20% of the time. A defective item is produced. What is the probability that it was produced by A?

99) An item is manufactured by three machines A, B and C. Out of the total number of items manufactured during a specified period, 50% are manufacture on machine A, 30% on B and 20% on C. 2% of the items produced on A and 2% of items produced on B are defective and 3% of these produced on C are defective. All the items stored at one godown. One item is drawn at random and is found to be defective. What is the probability that it was manufactured on machine A?

100) There are three coins. One is two-headed coin (having head on both faces), another is biased coin that comes up heads 75% of the times and third is also a biased coin that comes up tail 40% of the times. One of the three coins is chosen at random and tossed, and it shows up tail 40% of the times. One of the three coins is chosen at random and tossed, and it shows heads. What is the probability that it was the two-headed coin?

101) A company has two plants to manufacture bicycles. The first plant manufactures 60% of the bicycles and the second plant 40%. Out of the 80% of the bicycles are rated of standard quality at the first plant and 90% of standard quality at the second plant. A bicycle is picked up at random and found to be standard quality. Find the probability that it comes from the second plant.

102) Three urns A, B and C contain 6 red and 4 white; 2 red and 6 white; and 1 red and 5 white balls respectively. An urn is chosen at random and a ball is drawn. If the ball drawn is found to be red, find the probability that the ball was drawn from urn A.

103) In a group of 400 people, 160 are smokers and non-vegetarian, 100 are smokers and vegetarian and the remaining are non-smokers and non-vegetarian. The probabilities of getting a special chest disease are 35%, 20% and 10% respectively. A person is chosen from the group at random and is found to be suffering from the disease. What is the probability that the selected person is a smoker and non-vegetarian?

104) A factory has three machines A, B and C, which produce 100, 200 and 300 items of a particular type daily. The machines produce 2%, 3% and 5% defective items respectively. One day when the production was over, an item was found to be defective. Find the probability that it was produced by machine A.

105) A bag contains 1 white and 6 red balls, and a second bag contains 4 white and 3 red balls. One of the bags is picked up at random and a ball is randomly drawn from it, and is found to be white in colour. Find the probability that the drawn ball was from the first bag.

106) In a certain collage, 4% of boys and 1% of girls are taller than 1.75 meters. Furthermore, 60% of the students in the colleges are girls. A student selected at random from the college is found to be taller than 1.75 meters. Find the probability that the selected students is girl.

107) For A, B and C the chances of being selected as the manager of a firm are in the ratio 4:1:2 respectively. The respective probabilities for them to introduce a radical change in marketing strategy are 0.3, 0.8 and 0.5. If the change does take place, find the probability that it is due to the appointment of B or C.

108) An insurance company insured 2000 scooters and 3000 motorcycles. The probability of an accident involving a scooter is 0.01 and that of a motorcycle is 0.02. An insured vehicle met with an accident. Find the probability that the accidented vehicle was a motorcycle.

109) Of the students in a college, it is known that 60% reside in a hostel and 40% do not reside in hostel. Previous year results report that 30% of students residing in hotel attain A grade and 20% of ones not residing in hostel attain A grade in their annual examination. At the end of the year, one students is chosen at random from the college and he has an A grade. What is the probability that the selected student is a hosteler?

110) There are three coins, One is two headed coin, another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is choosen at random and tossed, it shows heads, it shows heads, what is the probability that it was the two headed coin?

111) Assume that the chances of a patient having a heart attack is 40%. It is also assumed that meditation and yoga course reduces the risk of heart attack by 30% and prescription of certain drug reduces its chances by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options and patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga?

112) If a machine is correctly set up it produces 90% acceptable items. If it incorrectly set up it produces only 40% acceptable

items. Past experience shows that 80% of the setups are correctly done. If after a certan set up, the machine produces 2 acceptable items, find the probability that the machine is correctly set up.

113) By examining the chest X-ray, probability that T.B. is detected when a person is actually suffering is 0.99. The probability that the doctor diagnoses incorrectly that a person has T.B. on the basis of X-ray is 0.001. In a certain city 1 in 1000 persons suffers from T.B. A person is selected at random is diagnosed to have T.B. What is the chance that he actually has T.B.

114) A is known to speak truth 3 times out of 5 times. He throws a die and reports that it is one. Find the probability that it is actually one.

115) A speaks the truth 8 times out of 10 times. A die is tossed. He reports that it was 5. What is the probability that it was actually 5?

116) A laboratory blood test is 99% effective in detecting a certain disease when its infection is present. However, the test also yields a false positive result for 0.5% of the healthy person tested (i.e. if a healthy person is tested, then with probability 0.005, the test will imply he has the disease). If 0.1% of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive?

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27 Mean and Variance of a Random Variable

Random Variable

A random variable is a real valued function, whose domain is the sample space of a random experiment. Generally it is denoted by X.

Probability distribution of random vareable

If a random variable X takes values $x_1, x_2, ...x_n$ with respective probabilities $p_1, p_2, ...p_n$, then

X	x_1	x_2	X_3	....	x_n
P(X)	p_1	p_2	p_3	....	p_n

Is known as the probability distribution of X.

1) Let X denote the number of hours you study during a randomly selected school day. The probability that X can take the value x has the following form, where k is some unknown constant.

$$P(X=x)=\begin{cases} 0.1 & ,\text{if } x=0 \\ kx & ,\text{if } x=1 \text{ or } 2 \\ 5(5-x) & ,\text{if } x=3 \text{ or } 4 \\ 0 & ,\text{otherwise} \end{cases}$$

(i) Find the value of k

What is the probability that you study (ii) At least two hours? (iii) Exactly two hours? (iv) At most two hours?

2) A rondom variable X has the following probability distribution:

X:	0	1	2	3	4	5	6	7
P(X) :	0	k	2k	3k	k^2	$2k^2$	$7k^2$	$7k^2 + k$

Find each of the following:

(i) k (ii) $P(X < 6)$

(iii) $P(X > 6)$ (iv) $P(0 < X < 5)$

3) Three cards are drawn from a pack of 52 playing cards. Find the probability distribution of the number of aces.

4) Four bad oranges are mixed accidently with 16 good orange. Find the probability distribution of the number of bad oranges in a draw of two oranges.

5) From a lot of 10 items containing 3 defectives, a sample of 4 items is drawn at random. Let the random variable X denote the number of defective items in the sample. If the sample is drawn randomly, find

(i) The probability distribution of X (ii) $P(X \leq 1)$

(iii) $P(X < 1)$ (iv) $P(0 < X < 2)$

6) A coin is biased so that the head is 3 times as likely to occur as tail. If coin is tossed twice, find the probability distribution for the number of tails.

7) Two biased dice are thrown together. For the first die $P(6)=\frac{1}{2}$, other scores being equally likely while for the second die, $P(1)=\frac{2}{5}$ and other scores are equally likely. Find the probability distribution of 'the number of one's seen.'

8) A class has 15 students whose ages are 14, 17, 15, 14, 21, 19, 20, 16, 18, 17, 20, 17, 16, 19 and 20 years respectively. One student is selected in such a manner that each has the same chance of being selected and the age X of the selected student

is recorded. Which is the probability distribution of the random variable X?

9) Two cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of aces.

10) Two cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of kings.

11) Two cards are drawn successively without replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of kings.

12) A fair die is tossed twice. If the number appearing on the top is less than 3, it is a success. Find the probability distribution of number of successes.

13) From a lot of 10 bulls, which includes 3 detectives, a sample of 2 bulbs is drawn at random. Find the probability distribution of the number of defective bulls.

14) Four balls are to be drawn without replacement from a box containing 8 red and 4 white balls. If X denotes the number of red balls drawn, find the probability distribution of X.

Mean, variance and standard deviation of a discrete random variable

Mean : If X is a discrete random variable which assumes $x_{1,} x_{2,} x_3 ....x_n$ with respective probabilities $p_{1,} p_{2,}, p_3 ....p_n$ then the mean of X is defined as

$$\bar{\bar{x}} = p_1 x_1 + p_2 x_2 + ...p_n x_n \quad \text{or} \quad \bar{\bar{x}} = \sum_{i=1}^{n} x_1 p_1$$

Variance: $\text{var}(x) = \sum_{i=1}^{n} x_1^{\,2} . px_i - \left(\sum_{i=1}^{n} x_1 p_1 \right)^2$

μ Or $\sigma^2 = E(x^2) - \{E(X)\}$

Or $V(X)=E(X-\mu)^2$

Standard deviation $\sigma_x = \sqrt{V(X)} = \sqrt{\sum_{i=1}^{n}(x_1-\mu)^2 px_1}$

15) If a pair of dice is thrown and X denotes the sum of the number on them. Find the probability distribution of X. Also, find the expectation of X.

16) A discret random variable X has the following probability distribution:

X	1	2	3	4	5	6	7
P(X)	c	2c	2c	3c	c^2	$2c^2$	$7c^2+c$

Find the value of c. Also, find the mean of the distribution.

17) The probability distribution of a random variable X is given below:

X:	0	1	2	3	4
P(X):	0.1	0.25	0.3	0.2	0.15

Find (i) Var (X) (ii) $\text{var}\left(\frac{X}{2}\right)$

18) The probability distribution of random variable X is given as

under: $P(X=x)=\begin{cases} kx^2 \text{ for } x=1,2,3 \\ 2kx \text{ for } x=4,5,6 \\ 0 \text{ otherwise} \end{cases}$,where k is a constant.

Find (i) $p(x \geq 4)$ (ii) E(x) (iii) $E(3X^2)$)

19) The probability distribution of the discrete random variables X and Y are given below:

X	0	1	2	3
P(X):	/5	/5	/5	/5

Y	0	1	2	3
P(Y):	/5	/10	/5	/10

Prove that : $E(Y^2) = 2E(X)$.

20) Find the mean $E(3X^2)$, variance and standard deviation of the number of heads in a simultaneous toss of three coins.

21) Two number are selected at random (without replacement) from the first six positive integers. Let X denotes the larger of the two number obtained. Find E(X) and Var (X).

22) In a meeting 70% of the members favour a certain proposal, 30% being opposed. A member is selected at random and let X = 0 if he opposed, and X = 1 if he is favour. Find E(X) and Var (X).

23) A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student is selected in such a manner that each has the same chance of being choosen and the age X of the selected student is recorded. What is the probability distribution of random variable X? Find mean, variance and standard deviation of X.

24) Find the probability distribution of the number of success in two tosses of a die, where a success is defined as 'getting a number greater than 4'. Also, find the mean and variance of the distribution.

25) Two cards are drawn successively with replacement from a well-shuffled deck of 52 cards. Find the mean and standard deviation of the number of aces.

26) Two cards are drawn successively without replacement from a well-shuffled deck of 52 cards. Computer the variance of the number of aces.

27) In a game a man wins a rupee for a six and losses a rupee for any other number when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a six. Find the expected value of the amount he wins/looses.

28) In a group of 30 scientists working on an experiment, 20 never error in their work and are reporting result elaborately. Two

scientists are selected at random from the group. Find the probability distribution of the number of selected scientists who never commit error in the work and reporting. Also, find the mean of the distribution. What value are described in the question?

29) A biased die is such that $P(4) = \frac{1}{10}$ and other scores being equally likely. The die is tossed twice. If X is the 'number of fours seens', find the variance of the random variable X.

30) Find the probability distribution of the maximum of two scores obtained when a die is thrown twice. Determine also the mean of the distribution.

31) Find the mean and standard deviation of each of the following probability distributions:

(i)

x_1:	2	3	4
p_1:	0.2	0.5	0.3

(ii)

x_1:	0	1	2	3	4	5
p_1:	1/6	5/18	2/9	1/6	1/9	1/18

32) A discrete random variable X has the probability distribution given below:

33) Find the mean and variance of the number of tails in three tosses of a coin.

34) Two cards are drawn simultaneously from a pack of 52 cards. Compute the man and standard deviation of the number of kings.

35) A box contains 13 bulbs, out of which 5 are defective. 3 bulbs are randomly drawn, one by one without replacement, from the box. Find the probability distribution of the number of defective bulbs.

36) Three cards are drawn at random (without replacement) from a well shuffled pack of 52 cards. Find the probability distribution of number of red cards. Hence find the mean of the distribution.

37) An urn contains 5 red 2 black balls. Two balls are randomly drawn, without replacement. Let X represent the number of black balls drawn. What are the possible values of X? Is X a random variable? If yes, find the mean and variance of X.

38) Two numbers are selected at random (without replacement) from position integers 2, 3, 4, 5, 6 and 7. Let X denote the larger of the two numbers obtained. Find the mean and variance of the probability distribution of X.

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28 Binominal Distribution

Bernoulli trials

Trials of a random experiment are called Bernoulli trials if they satisfies the following condition:

(i) There should be a finite number of trials.

(ii) The trial should be independent.

(iii) Each trial has exactly two outcomes, success or failure.

(iv) The probability of success or failure remains same in each trial.

Binomial distribution

Let x be a random variable which can take n values 0,1,2,……,n. then by binomial distribution , we have

$p(x=r) = {}^nC_r\, p^r q^{n-r}$ where,

n = Total number of trials in an experiments,

p = Probability of success in one trial

q = Probability of failure in one trial

r = Number of success trial in an experiment.

Also, $p+q=1$.

Binomial distribution of the number of success X can be represented as

X	0	1	2	3	….r..	n
P(X)	${}^nC_0p^0q^n$	${}^nC_0p^0q^{n-1}$	${}^nC_0p^0q^{n-2}$	${}^nC_0p^0q^{n-3}$	${}^nC_0p^0q^{n-r}$	${}^nC_np^n$

1) A pair of dice is thrown 7 times. If getting a total of 7 is considered a success, what is the probability of

(i) no success? (ii) 6 successes? (iii) at least 6 successes? (iv) at most 6 successes?

2) A man takes a step forward with probability 0.4 and backwards with probability 0.6. Find the probability that at the end of eleven steps he is just one step away from the starting point.
3) In a hurdles race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\frac{5}{6}$. What is the probability that he will knock down than 2 hurdles?
4) An unbiased die is thrown again and again until three sixes are obtained. Find the probability of obtaining 3^{rd} six in the sixth throw of the die.
5) For 6 trails of an experiment, let x be a binomial variate which satisfies the relation $9\,p(x=4) = p(x=2)$. Find the probability of success.
6) Find the probability distribution of the number of doublets in 4 throws of a pair of dice.
7) Five cards are drawn are drawn successively with replacement from a well shuffled pack of 52 cards. What is the probability that (i) all the five cards are spades? (ii) only 3 cards are spades? (iii) none is spades?
8) Suppose that 90% people are right-handed. What is the probability that at most 6 of a random sample of 10 people are right handed?
9) Find the probability distribution of the number of doublets in 4 throws of a pair of dice. Also, find the mean and variance of this distribution.
10) Find the probability distribution of the number of sixes in three tosses of a die.
11) An unbiased coin is tossed 8 times. Find, by using binomial distribution, the probability of getting at least 6 heads.
12) Six coins are tossed simultaneously. Find the probability of getting (i) 3 heads (ii) no heads (iii) at least one head

13) An experiment succeeds twice as often as it fails. Find the probability that in the next 6 trials there will be at least 4 successes.

14) The probability that a student entering a university will graduate is 0.4. Find the probability that out of 3 students of the university:

    (i) None will graduate, (ii) only one will graduate,

    (iii) all will graduate.

15) Ten eggs are drawn successively, with replacement, from a lot containing 10% defective eggs. Find the probability that there is at least one defective egg.

16) In a 20- question true-false examination suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answer 'true'; if it falls tails, he answers 'false'. Find the probability that he answer at least 12 questions correctly.

17) Suppose x has a binomial distribution with $n=6$ and $p=\frac{1}{2}$. Show that $x=3$ is the most likely outcome.

18) In a multiple choice examination with three possible answers for each of the five questions out of which only one is correct, what is the probability that a candidate would get four or more correct answer just by guessing?

19) A person buys a lottery ticket in 50 lotteries, in each of which his chance of winning a prize is $\frac{1}{100}$. What is the probability that he will win a prize?

    (i) At least one (ii) exactly only (iii) at least twice?

20) The probability of a shooter hitting a target is $\frac{3}{4}$. How many minimum numbers of times must he/ she fire so that the probability of hitting the target at least once is more than 0.99?

21) How many times must a man toss a fair coin so that the probability of having at least one head is more than 90%?

22) How many times must a man toss a fair coin so that the probability of having at least one head is more than 80%?

23) A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability distribution of the number of successes.

24) From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

25) Find the probability that in 10 throws of a fair die a score which is a multiple of 3 will be obtained in at least 8 of the throws.

26) A die is thrown 5 times. Find the probability that an odd number will come up exactly three times.

27) The probability of a man hitting a target is 0.25. He shoots 7 times. What is the probability his hitting at least twice?

28) A factory produces bulbs. The probability that one bulb is defective is $\frac{1}{50}$ and they are packed in boxes of 10. From a single box, find the probability that

(i) None of the bulbs is defective

(ii) exactly two bulbs are defective

(iii) More than 8 bulbs work properly.

Mean and variance of binomial distribution

(A) Mean $(\mu) = \sum x_i p_i = np$

(B) Variance $(\sigma^2) = \sum_{i=1}^{n} x_i^2 - \mu^2 = npq$

(C) Standard deviation (σ) = $\sqrt{Variance} = \sqrt{npq}$

Note: mean is always greater than variance.

Practice problem

29) The mean and variance of a binomial distribution are 4 and 4/3 respectively; find $P(x \geq 1)$.

30) If the sum of the mean and variance of a binomial distribution for 5 trails is 1.8, find the distribution.

31) If two dice are rolled 12 times, obtain the mean and the variance of the distribution of successes, if getting a total greater than 4 is considered a success.

32) If a random variable X follows binomial distribution with mean 3 and variance 3/2, find $p(x \leq 5)$.

33) If x follows binomial distribution with mean 4 and variance 2, find $p(x \geq 5)$.

34) The mean and variance of a binomial distribution are $\frac{4}{3}$ and $\frac{8}{9}$ respectively. Find $p(x \geq 1)$

35) If the sum of the mean and variance of a binomial distribution for 6 trials is , find the distribution.

36) A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability distribution of number of successes and hence find its mean.

37) Find the probability distribution of the number of doublets in three throws of a pair of dice and hence find its mean.

38) Form a lot of 15 bulbs which include 5 defective, a sample of 4 bulbs is drawn one by one with replacement. Find the probability distribution of number of defective bulbs. Hence find the mean of the distribution.

39) A die is thrown three times. Let x be 'the number of two seen'. Find the expectation of x.

40) A die is tossed twice. 'A' success' is getting an even number on a toss. Find the variance of number of successes.

41) Three cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of spades. Hence, find the mean of the distribution.

42) An unbiased coin is tossed 4 times. Find the mean and variance of the mean and variance of the number of heads obtained.

43) If x follows binomial distribution with parameters $n = 5, p \ and \ p(x = 2) = 9p(x = 3)$, them find the value of p.

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